Exercises for Commutative Algebra
List 5 to hand in at 17.4.2017

Exercise 1.
Let A be an integral domain with field of fractions K and M an A-module. The *torsion submodule of* M is the set $T(M) = \{m \in M | \operatorname{Ann}(m) \neq \{0\}\}$.

1. Show that $T(M)$ is equal to the kernel of the A-linear map $M \to M \otimes_A K$, sending m to $m \otimes 1$, and conclude that $T(M)$ is an A-submodule of M.

   Hint: Use that $M \otimes_A K \simeq S^{-1}M$ for $S = A - \{0\}$.

2. Show that $\overline{M} = M/T(M)$ is *torsion-free*, i.e. $T(\overline{M}) = 0$.

3. Let $f : M \to N$ be an A-linear map. Show that $f(T(M)) \subset T(N)$.

4. An *torsion module over* A is an A-module M such that $T(M) = M$. Define $A - \operatorname{Mod}^t$ as the category of torsion A-modules together with A-linear maps and define $T(f) = f|_{T(M)} : T(M) \to T(N)$. Show that this defines a functor $T : \mathbb{A} - \operatorname{Mod} \to A - \operatorname{Mod}^t$.

5. Show that T is right adjoint to the inclusion $\iota : A - \operatorname{Mod}^t \to A - \operatorname{Mod}$ as a subcategory and conclude that T is left exact.

Exercise 2.
Let A be a ring and $f : N \to M$ and $g : M \to P$ be A-linear maps. Show that the following are equivalent.

1. $N \to M \to P$ is exact at M.

2. $N_\mathfrak{p} \to M_\mathfrak{p} \to P_\mathfrak{p}$ is exact at $M_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of A.

3. $N_\mathfrak{m} \to M_\mathfrak{m} \to P_\mathfrak{m}$ is exact at $M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of A.

Exercise 3.
Let $f : A \to B$ be a ring homomorphism, M an A-module and N a B-module. Show that the extension of scalars $f_*(M) = M \otimes_A B$ and the restriction of scalars $f^*(N) = N$ (considered as A-module) extend to functors $f_* : A - \operatorname{Mod} \to B - \operatorname{Mod}$ and $f^* : B - \operatorname{Mod} \to A - \operatorname{Mod}$ such that f_* is left-adjoint to f^*.

Exercise 4.
Let A be a principal ideal domain. Show that every ideal of A has a primary decomposition.

Exercise 5.
Let k be a field and $A = k[x, y, z]$. Consider the prime ideals $\mathfrak{p} = (x, y)$ and $\mathfrak{q} = (x, z)$ and the maximal ideal $\mathfrak{m} = (x, y, z)$. Show that $I = \mathfrak{p} \cap \mathfrak{q} \cap \mathfrak{m}^2$ is a reduced primary decomposition of $I = \mathfrak{p} \cdot \mathfrak{q}$. Which components are isolated and which are embedded? Make an illustration of the affine variety $\widetilde{V}(I)$ in $\mathbb{A}^3_k$ and the respective (irreducible and embedded) components.

Here are some additional exercises from Atiyah-Macdonald (which are not to hand in): chapter 3, exercises 13, 19 and 21; chapter 4, exercises 10 and 11.