Exercises for Algebra II
List 12 Preparation list for the exam - not to hand in

Exercise 1.
Show that an element $a \in \overline{\mathbb{Q}}$ is an algebraic integer if and only if the the subring $\mathbb{Z}[a]$ of $\overline{\mathbb{Q}}$ is finitely generated as a $\mathbb{Z}$-module. Show that in this case, $\mathbb{Z}[a]$ is a free $\mathbb{Z}$-module.

Exercise 2.
Describe and prove the universal property for Ind_H^G.

Exercise 3.
Calculate for the following finite groups $H < G$ and for every simple character χ of H the decomposition of $\mathrm{Ind}_H^G \chi$ into simple characters of G, using the character tables of both H and G.

1. $H = A_3$ and $G = S_3$.
2. $H = V$ (Klein 4-group) and $G = S_4$.
3. $H = A_4$ and $G = S_4$.

Hint: Use Frobenius reciprocity to facilitate the calculations.

Remark: Note that the multiplicities occuring in these decompositions can be organized in a table (the *induction-restriction table for H and G*) as follows:

	ψ_1	$\cdots$	ψ_r
χ_1	$\langle \mathrm{Ind}\,\chi_1, \psi_1 \rangle_G$	$\cdots$	$\langle \mathrm{Ind}\,\chi_1, \psi_r \rangle_G$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_s	$\langle \mathrm{Ind}\,\chi_s, \psi_1 \rangle_G$	$\cdots$	$\langle \mathrm{Ind}\,\chi_s, \psi_r \rangle_G$

Exercise 4.
Let G be a finite group, $g \in c$ and $C_G(g) = \{h \in G | gh = hg\}$ the *centralizer of g in G*. Let c be the conjugacy class of g. Show that $\#c = \frac{\#G}{\#C_G(g)}$ and conclude that, in particular, $\#c$ is a divisor of $\#G$.

Exercise 5.
Let G be a non-abelian group of order 55.

1. Show that G is solvable. More precisely, show that G has a normal subgroup N of order 11. Conclude that $G^{\mathrm{ab}} = G/N$.
2. Show that G has precisely 7 simple characters and determine their dimensions.
3. Show that G has four conjugacy classes with 11 elements, two conjugacy classes with 5 elements and one conjugacy class with 1 element.
4. Determine the character table of G.

Exercise 6.
Let $G = \{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}(\mathbb{F}_7) \mid a = c^2 \text{ for some } c \in \mathbb{F}_7^\times \}$.

1. Show that G is a subgroup of $\mathrm{GL}(\mathbb{F}_7)$ of order 21.
2. Show that $N = \{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \}$ is a normal subgroup of G and that $G^{\mathrm{ab}} = G/N$.
3. Show that G has precisely 5 simple characters, three of dimension 1 and two of dimension 3.
4. Determine the conjugacy classes $c_1, \dots, c_5$ of G and their cardinalities.
5. Determine the character table of G.
6. Let $G' = \{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}(\mathbb{F}_7) \mid a \in \mathbb{F}_7^\times \}$. Determine the conjugacy classes of G' and exhibit a set S of representatives for these conjugacy classes.
7. Use Theorem 3.3.1 of the lecture to compute $\mathrm{Ind}_G^{G'} \chi(g')$ for every simple character χ of G and every $g' \in S$.
8. Use Mackey's criterion to verify for which simple characters χ of G the induced character $\mathrm{Ind}_G^{G'} \chi$ is simple.

Exercise 7.
Determine the character table for the dihedral groups D_5 and D_6.

Exercise 8.
Choose your five favourite character tables from https://people.maths.bris.ac.uk/~matyd/GroupNames/characters.html and verify that they are correct.