Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
List 1 To hand in at 17.3.2020 in class

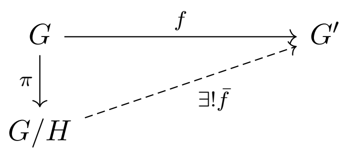
Exercise 1.
Let G be a commutative group and H a subset. Show that H is a subgroup if and only if the multiplication μ of G restricts to a map $\mu_H : H \times H \to H$ such that (H, μ_H) is a commutative group.

Exercise 2 (Group homomorphisms)**.**
Let G and H be commutative groups. A *group homomorphism between G and H* is a map $f : G \to H$ such that $f(ab) = f(a)f(b)$ for all $a, b \in G$.

1. Let $f : G \to H$ be a group homomorphism. Show that $f(e_G) = e_H$ and $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$ where e_G is the neutral elements of G and e_H is the neutral element of H.

2. Show that the identity map $\mathrm{id} : G \to G$ is a group homomorphism and that the composition $g \circ f : G \to H'$ of two group homomorphisms $f : G \to H$ and $g : H \to H'$ is a group homomorphism.

Exercise 3 (Universal property of quotient groups)**.**
Let H be a subgroup of a commutative group G and G/H the quotient. Show that the association $a \mapsto [a]$ defines a group homomorphism $\pi : G \to G/H$ with $\pi(H) = \{0\}$. Show that for every group homomorphism $f : G \to G'$ with $f(H) = \{0\}$, there is a unique group homomorphism $\bar{f} : G/H \to G'$ such that $f = \bar{f} \circ \pi$:



Exercise 4 (Cyclic groups)**.**
Let G be a commutative group and $a \in G$. We define

$$a^n = \underbrace{a \cdots a}_{n\text{-times}} \quad \text{for } n > 0, \qquad a^0 = e, \quad \text{and} \qquad a^n = \underbrace{a^{-1} \cdots a^{-1}}_{-n\text{-times}} \quad \text{for } n < 0.$$

We call G a *cyclic group* if there is an element $a \in G$ such that every other element $b \in G$ is of the form $b = a^n$ for some $n \in \mathbb{Z}$.

1. Show that there is a cyclic group C_n for every $n \geq 1$.

2. Are there infinite cyclic groups?

3. Given two commutative groups G and H, we define their product as the Cartesian product $G \times H = \{(g, h) | g \in G, h \in H\}$, together with the componentwise multiplication, i.e. $(g, h) \cdot (g', h') = (gg', hh')$. Show that $G \times H$ is a commutative group.

4. Show that $C_n \times C_m$ is cyclic if the greatest common divisor of m and n is 1.

5. Is $C_2 \times C_2$ cyclic?