Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
List 10 To be sent to xia.xiao@impa.br until 22.5.2020

Exercise 1.
Let A be a ring and $f : M \to N$ a homomorphism of A-modules. Show that

1. f is a monomorphism if and only if it is injective;
2. f is an epimorphism if and only if it is surjective;
3. f is an isomorphism (in the sense of category theory, cf. Chapter 2) if and only if it is bijective.

Exercise 2. Show that the following properties for an A-module P are equivalent.

1. The functor $\mathrm{Hom}(P, -)$ is exact.
2. There is an A-module Q such that $P \oplus Q$ is free.
3. Every short exact sequence of A-modules of the form $0 \to N \to M \to P \to 0$ splits.
4. For every epimorphism $p : M \to Q$ of A-modules and every homomorphism $f : P \to Q$, there is a homomorphism $g : P \to M$ such that $f = p \circ g$.

An A-module P with these properties is called *projective*. Conclude that every free A-module is projective. Show that $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module that is not free.

Remark: An A-module I is *injective* if $\mathrm{Hom}(-, I)$ is exact. It can be shown that there are analogous characterizations as in (3) and (4) for injective modules. However, there is no direct analogue to (2). For $A = \mathbb{Z}$, one can show that a $\mathbb{Z}$-module I is injective if and only if it is *divisible*, i.e. for every $m \in I$ and every integer $l > 0$ there exists an $n \in I$ such that $l.n = m$.

Exercise 3.
Let A be a ring and P an A-module.

1. Let M and N be A-modules and $f : N \to \mathrm{Hom}_A(P, N)$ a homomorphism. Show that $\Psi_{M,N}(f) : p \otimes m \mapsto \big(f(m)\big)(p)$ defines an isomorphism
$$\Psi_{M,N} : \quad \mathrm{Hom}_A(P \otimes_A M, N) \quad \longrightarrow \quad \mathrm{Hom}_A\big(M, \mathrm{Hom}_A(P, N)\big)$$
of A-modules.
2. Let $\alpha : M \to M'$ and $\beta : N \to N'$ be homomorphisms. Show that the diagram
$$\begin{array}{ccc} \mathrm{Hom}_A\big(M', \mathrm{Hom}_A(P, N)\big) & \xrightarrow{\Psi_{M',N}} & \mathrm{Hom}_A(P \otimes_A M', N) \\ {\scriptstyle \beta_* \circ - \circ \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \beta \circ - \circ \alpha_P} \\ \mathrm{Hom}_A\big(M, \mathrm{Hom}_A(P, N')\big) & \xrightarrow{\Psi_{M,N'}} & \mathrm{Hom}_A(P \otimes_A M, N') \end{array}$$
commutes where $\alpha_P : P \otimes_A M \to P \otimes_A M'$ and $\beta_* : \mathrm{Hom}_A(P, N) \to \mathrm{Hom}_A(P, N')$ are the homomorphisms that are induced by α and β, respectively.

Exercise 4.
Let A be a ring and M_1 and M_2 A-modules.

1. Show that the canonical injections $\iota_k : M_k \to M_1 \oplus M_2$ and the canonical projections $\pi_k : M_1 \oplus M_2 \to M_k$ (for $k = 1, 2$) satisfy the relations

$$\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 \; = \; \mathrm{id}_{M_1 \oplus M_2} \qquad \text{and} \qquad \pi_k \circ \iota_l \; = \; \begin{cases} \mathrm{id}_{M_k} & \text{if } k = l, \\ \mathbf{0} & \text{if } k \neq l \end{cases}$$

   for all $k, l = 1, 2$.

2. Let P be an A-module and $i_k : M_k \to P$ and $p_k : P \to M_k$ homomorphisms for $k = 1, 2$ that satisfy the relations

$$i_1 \circ p_1 + i_2 \circ p_2 \; = \; \mathrm{id}_P \qquad \text{and} \qquad p_k \circ i_l \; = \; \begin{cases} \mathrm{id}_{M_k} & \text{if } k = l, \\ \mathbf{0} & \text{if } k \neq l \end{cases}$$

   for $k, l = 1, 2$. Show that the homomorphism $M_1 \oplus M_2 \to P$ that is induced by $\{i_k : M_k \to P\}_{k=1,2}$ is an isomorphism.

3. Let B be a ring and $\mathcal{F} : \mathrm{Mod}_A \to \mathrm{Mod}_B$ an additive covariant functor. Show that the homomorphism $\mathcal{F}(M_1) \oplus \mathcal{F}(M_2) \to \mathcal{F}(M_1 \oplus M_2)$ that is induced by $\{\mathcal{F}(\iota_k) : \mathcal{F}(M_i) \to \mathcal{F}(M_1 \oplus M_2)\}_{k=1,2}$ is an isomorphism.

4. Let $\mathcal{F} : \mathrm{Mod}_A \to \mathrm{Mod}_B$ be as before and $0 \to N \to M \to Q \to 0$ a split short exact sequence. Show that $0 \to \mathcal{F}(N) \to \mathcal{F}(M) \to \mathcal{F}(Q) \to 0$ is a split short exact sequence.

Exercise 5 (Bonus)**.**
Let A and B be rings.

1. Let M and N be A-modules and $f, g : M \to N$ homomorphisms. Show that the homomorphism

$$\begin{array}{ccccccc} M & \xrightarrow{\Delta} & M \oplus M & \xrightarrow{(f,g)} & N \oplus N & \xrightarrow{\Sigma} & N \\ m & \longmapsto & (m, m) & & (m, n) & \longmapsto & m + n \\ & & (m, n) & \longmapsto & f(m) + g(n) & & \end{array}$$

   is equal to $f + g : M \to N$.

2. Let $\mathcal{F} : \mathrm{Mod}_A \to \mathrm{Mod}_B$ be a covariant functor such that for all A-modules M_1 and M_2, the homomorphism $\mathcal{F}(M_1) \oplus \mathcal{F}(M_2) \to \mathcal{F}(M_1 \oplus M_2)$ that is induced by $\{\mathcal{F}(\iota_k) : \mathcal{F}(M_i) \to \mathcal{F}(M_1 \oplus M_2)\}_{k=1,2}$ is an isomorphism where $\iota_k : M_k \to M_1 \oplus M_2$ are the canonical inclusions. Show that $\mathcal{F}$ is additive.