Exercise 1.
Let K be a field and $A = K[T]$ and consider $M = K^n$ as an A-module by letting T act as a complex $n \times n$-matrix U. Show that M is a cyclic A-module if U has a Jordan normal form with only one Jordan block, i.e. if U is conjugated to a matrix of the form

$$\begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}$$

for some $\lambda \in K$.

Exercise 2. Consider the $\mathbb{C}[T]$-module $M = \mathbb{C}^3$ where T acts as one of the matrices

$$(1)\quad T = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad (2)\quad T = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad (3)\quad T = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}$$

$$(4)\quad T = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad (5)\quad T = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad (6)\quad T = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

and where λ, μ and ν are pairwise distinct complex numbers. Determine in each case the characteristic polynomial and the minimal polynomial of T, as well as the elementary divisors and the invariant factors of M.

Exercise 3.
Let K be a field, M a finite dimensional K-vector space and $\varphi : M \to M$ a K-linear map. Let $I_1 = (f_1), \ldots, I_s = (f_s)$ be the invariant factors of M as $K[T]$-module where T acts as φ and where $f_1, \ldots, f_s$ are monic polynomials.

1. Show that $\prod_{i=1}^{s} f_i$ is the characteristic polynomial of φ.

   Hint: Reduce the situation to the case where M is cyclic and use that in this case, the characteristic polynomial equals the minimal polynomial.

2. The K-linear map $\varphi : M \to M$ is called *diagonalizable* if it acts as a diagonal matrix with respect to some basis of M. Show that φ is diagonalizable if and only if the minimal polynomial is of the form

   $$\mathrm{Min}_\varphi = \prod_{i=1}^{n} (T - \alpha_i)$$

   for pairwise distinct $\alpha_1, \ldots, \alpha_n \in K$. Is the $\mathbb{C}$-linear map $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ given by the matrix $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ for the standard basis of $\mathbb{C}^2$ diagonalizable?

Exercise 4.
Let A be a ring and $\mathrm{Mat}_{n\times n}(A)$ the set of $n \times n$-matrices with coefficients in A.

1. Show that $\mathrm{Mat}_{n\times n}(A)$ is a noncommutative ring with respect to matrix addition and matrix multiplication. What are 0 and 1?

2. Show that the inclusion $f : A \to \mathrm{Mat}_{n\times n}(A)$ as diagonal matrices is a homomorphism of (noncommutative) rings, i.e. $f(a+b) = f(a) + f(b)$, $f(a \cdot b) = f(a) \cdot f(b)$ and $f(1) = 1$.

3. The *determinant* is the map $\det : \mathrm{Mat}_{n\times n}(A) \to A$ that sends a matrix $T = (a_{i,j})_{i,j=1,\dots,n}$ to the element

$$\det(T) \quad = \quad \sum_{\sigma\in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

   of A. Show that det is multiplicative, i.e. $\det(T \cdot T') = \det(T) \cdot \det(T')$ and $\det(1) = 1$.

4. Show that a matrix T is a unit in $\mathrm{Mat}_{n\times n}(A)$, i.e. $TT' = 1$ for some matrix T', if and only if $\det(T)$ is a unit in A.

Exercise 5 (Bonus)**.**
Let K be a field, M and N finite dimensional K-vector spaces, and $\varphi : M \to M$ and $\psi : N \to N$ K-linear maps. Assume that their respective characteristic polynomials factor as

$$\mathrm{Char}_\varphi \; = \; \prod_{i=1}^{m} (T - \alpha_i), \text{ and } \quad \mathrm{Char}_\psi \; = \; \prod_{j=1}^{n} (T - \beta_j).$$

Show that the formula $\varphi \otimes \psi(m \otimes n) = \varphi(m) \otimes \psi(n)$ defines a K-linear homomorphism $\varphi \otimes \psi : M \otimes_K N \to M \otimes_K N$, whose characteristic polynomial is

$$\mathrm{Char}_{\varphi\otimes\psi} \; = \; \prod_{i,j} (T - \alpha_i\beta_j).$$