Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
List 15 To be sent to xia.xiao@impa.br until 26.5.2020

Exercise 1.
Consider the group homomorphism $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by the matrix

$$\begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

Determine the Smith normal form of φ and a decomposition of $\mathbb{Z}^3/\mathrm{im}\varphi$ into cyclic and indecomposable factors.

Exercise 2.
Let k be a field and $M = k^3$ be the $k[T]$-module where T acts as the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1. Calculate the characteristic polynomial and the minimal polynomial of this matrix, and determine the invariant factors and the elementary divisors of M.
2. Let $N = k$ be $k[T]$-module where T acts as 1. Show that there is a $k[T]$-module P and a split exact sequence of the form
$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} P \longrightarrow 0.$$
3. Show that $M \otimes_{k[T]} N \simeq N$ and that the sequence
$$0 \longrightarrow N \otimes_{k[T]} N \xrightarrow{\iota_N} M \otimes_{k[T]} N \xrightarrow{\pi_N} P \otimes_{k[T]} N \longrightarrow 0$$
is exact.

Exercise 3.
Let A and B be rings with 6 elements. Show that there is a unique ring isomorphism $A \to B$.

Exercise 4.
Consider the plane affine curve C given by $f = X^2 - XY + Y^2 - X - Y + 1$.

1. Find all singular points of C.
2. Show that $\mathcal{O}(C)$ is not a principal ideal domain.

Exercise 5.
Let k be a field and M a k-vector space with basis $\{b_1, b_2, b_3, b_4\}$. Given elements $m = \sum_{i=1}^4 x_i.b_i$ and $n = \sum_{i=1}^4 y_i.b_i$ of M, we can write $m \wedge n \in \Lambda^2 M$ as $\sum_{1\le i<j\le 4} \Delta_{i,j}.b_i \wedge b_j$ with $\Delta_{i,j} \in k$.

1. Show that $\Delta_{i,j} = x_i y_j - x_j y_i$ for all i, j with $1 \le i < j \le 4$.

2. Show that the $\Delta_{i,j}$ satisfy the so-called *Plücker relation*

$$\Delta_{1,2}\Delta_{3,4} - \Delta_{1,3}\Delta_{2,4} + \Delta_{1,4}\Delta_{2,3} \;=\; 0.$$

3. Show, conversely, that an arbitrary element $\sum_{1\le i<j\le 4} \Delta_{i,j}.b_i \wedge b_j$ of $\Lambda^2 M$ is of the form $m \wedge n$ for some $m, n \in M$ if the $\Delta_{i,j}$ satisfy the Plücker relation.

   Hint: Determine the $\Delta_{i,j}$ for $m = 1.b_1 + x_3.b_3 + x_4.b_4$ and $n = 1.b_2 + y_3.b_3 + y_4.b_4$. Use this to treat the case where $\Delta_{1,2} = 1$. Reduce the general situation to this case.