Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
List 3 To be sent to xia.xiao@impa.br until 30.3.2020

Exercise 1. Let A be a ring and $\{I_i\}_{i\in I}$ be a family of ideals in A.

1. Show that the intersection $\bigcap_{i\in I} I_i$ is an ideal of A.
2. For finite I, show that the product $\prod_{i\in I} I_i = \big(\{\prod a_i | a_i \in I_i\}\big)$ is an ideal of A that is contained in $\bigcap_{i\in I} I_i$. Under which assumption is $\prod I_i = \bigcap I_i$?
3. For finite I, show that $\sum I_i$ is indeed an ideal.
4. Let $f : A \to B$ be a ring homomorphism and I an ideal of B. Show that $f^{-1}(I)$ is an ideal of A. Show that $f^{-1}(I)$ is prime if I is prime. Is $f^{-1}(I)$ maximal if I is maximal? Is the image $f(J)$ of an ideal J of A an ideal of B?

Exercise 2.

1. Describe all ideals of $\mathbb{Z}$. Which of them are principal ideals, which of them are prime and which of them are maximal?
2. Let $f \in \mathbb{R}[T]$ be of degree ≤ 2. When is (f) a prime ideal, when is it a maximal ideal? When is the quotient ring isomorphic to $\mathbb{R}$? When is it isomorphic to $\mathbb{C}$?

Exercise 3.
Let $e_1, \ldots, e_n$ be pairwise coprime positive integers. Show that the underlying additive group of $\mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_n\mathbb{Z}$ is a cyclic group.

Exercise 4.
Let A be an integral domain and $a, b, c, d, e \in A$.

1. Show that if d is a greatest common divisor of b and c and e is a greatest common divisor of ab and ac, then $(e) = (ad)$. Conclude that $\gcd(ab, ac) = (a) \cdot \gcd(b, c)$.
2. If A is a principal ideal domain, then d is a greatest common divisor of a and b if and only if $(a, b) = (d)$. Conclude that every two elements of a principal ideal domain have a greatest common divisor.
3. Find an integral domain A with elements $a, b, d \in A$ such that d is a greatest common divisor of a and b, but $(a, b) \neq (d)$.

Exercise 5 (Bonus exercise)**.**
Show that the ring $\mathbb{Z}$ satisfies the following universal property: for every ring A, there is a unique ring homomorphism $f : \mathbb{Z} \to A$. Use this and the universal property of the quotient map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ to show that for a ring A whose underlying additive group $(A, +)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z}, +)$ (as a group), there is a unique ring isomorphism $\mathbb{Z}/n\mathbb{Z} \to A$.
Remark: We say that $\mathbb{Z}$ is an *initial object in the category of rings.*

*__Exercise 6__ (Bonus exercise)**.** [1]
Show that the ring $\mathbb{Z}[T]/\langle T^2 - T + 5\rangle$ is a principal ideal domain but not a Euclidean domain. This can be done along the following steps.

1. Reason that $N(a + bT) = a^2 + ab + 5b^2$ defines a map $N : \mathbb{Z}[T]/\langle T^2 - T + 5\rangle \to \mathbb{N}$. Show that $N(0) = 0$, $N(1) = 1$ and $N(xy) = N(x)N(y)$. Use the map N to show that the units of $\mathbb{Z}[T]/\langle T^2 - T + 5\rangle$ are ± 1 and that 2 and 3 are irreducible.

   Note: an element a is *irreducible*, if it is not zero nor a unit and if $a = bc$ implies that either b or c is a unit.

2. Show that every Euclidean domain A that is not a field contains an element $a \notin A^\times \cup \{0\}$ such for every $b \in A$ there is an element $u \in A^\times \cup \{0\}$ such that $a|(b - u)$. Conclude that $\mathbb{Z}[T]/\langle T^2 - T + 5\rangle$ is not a Euclidean domain, by considering the cases $a = 2$ and $a = T$.

3. Show that N is a *Dedekind-Hasse norm*, i.e. for all nonzero $x, y \in \mathbb{Z}[T]/\langle T^2 - T + 5\rangle$ such that y does not divide x, there are $p, q, r \in \mathbb{Z}[T]/\langle T^2 - T + 5\rangle$ such that $px = qy + r$ and $N(r) < N(y)$. Conclude that $\mathbb{Z}[T]/\langle T^2 - T + 5\rangle$ is a principal ideal domain.

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[1] Starred exercises are hard problem for those of you that search for a challenge. To balance the amount of work required to solve these exercises, starred exercises they are worth twice as many points as normal exercises.