---

**Exercise 1.**

1. Determine all units, prime elements and irreducible elements of $\mathbb{Z}/6\mathbb{Z}$.
2. Let $\mathbb{R}[T_1, T_2] = \big(\mathbb{R}[T_1]\big)[T_2]$ be the polynomial ring over $\mathbb{R}$ in $T_1$ and $T_2$ and $I$ the ideal generated by $T_1^2 + T_2^2$. Is the class $\overline{T}_1 = T_1 + I$ a prime element in the quotient ring $\mathbb{R}[T_1, T_2]/I$? Is $\overline{T}_1$ irreducible?

**Exercise 2.**
Let $\mathbb{Z}[\sqrt{-5}]$ be the set of complex numbers of the form $z = a + b\sqrt{-5}$ with $a, b \in \mathbb{Z}$ and $\sqrt{-5} = i\sqrt{5}$.

1. Show that $\mathbb{Z}[\sqrt{-5}]$ is a subring of $\mathbb{C}$.
2. Show that the association $a + b\sqrt{-5} \mapsto a^2 + 5b^2$ defines a map $N : \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}$ with $N(zz') = N(z)N(z')$ and $N(1) = 1$.
   **Remark:** $N(z)$ is the square of the usual absolute value of the complex number $z$.
3. Conclude that $z \in \mathbb{Z}[\sqrt{-5}]^\times$ if and only if $N(z) \in \mathbb{Z}^\times$. Determine $\mathbb{Z}[\sqrt{-5}]^\times$.
4. Show that 2, 3, $(1 + \sqrt{-5})$ and $(1 - \sqrt{-5})$ are irreducible, but not prime.
5. Show that 6 and $2 + 2\sqrt{-5}$ do not have a greatest common divisor.

**Exercise 3.**
Let $A$ be a unique factorization domain.

1. Show that every prime ideal of $A$ is generated by a set of prime elements.
2. Find an example of a unique factorization domain $A$ and prime elements $p_1, \ldots, p_n$ of $A$ such that $I = (p_1, \ldots, p_n)$ is **not** a prime ideal.
3. Show that the ideal $I = (2, 1 + \sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$ is prime and that it does not contain any prime element.
   *Hint:* Show that $\mathbb{Z}[\sqrt{-5}]/\langle a \rangle$ is finite for every nonzero $a \in I$ and use that finite integral domains are fields to deduce a contradiction to the assumption that $a$ is prime.

**Exercise 4.**
Let $A$ be an integral domain and $(a)$ a nonzero principal ideal of $A$. A *factorization of $(a)$ into principal prime ideals* is an equality of the form $(a) = \prod_{i=1}^{n} (p_i)$ where $(p_i)$ are principal prime ideals of $A$.

1. Show that a factorization in principal prime ideals is unique, i.e. if $(a) = \prod_{i=1}^{n} (p_i)$ and $(a) = \prod_{j=1}^{m} (q_j)$ are two such factorizations, then there exists a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $(p_i) = (q_{\sigma(i)})$ for all $i = 1, \ldots, n$.
2. Show that $A$ is a unique factorization domain if and only if every principal ideal of $A$ has a factorization into principal prime ideals.

**Exercise 5** (Bonus exercise)**.**
Let $A$ be a ring and $I$ an ideal of $A$. Show that $I$ is contained in a maximal ideal of $A$.
**Hint:** This is a consequence of Zorn's Lemma.

**Exercise 6** (Bonus exercise)**.**
Let $A$ be a ring. The *spectrum of* $A$ is the set $\operatorname{Spec} A$ of all prime ideals of $A$. A *principal open subset of* $\operatorname{Spec} A$ is a subset of the form

$$U_a \;=\; U_{A,a} \;=\; \{\, \mathfrak{p} \in \operatorname{Spec} A \,|\, a \notin \mathfrak{p} \,\}$$

with $a \in A$.

1. Show that $U_0 = \emptyset$, $U_1 = \operatorname{Spec} A$ and $U_a \cap U_b = U_{ab}$ for all $a, b \in A$.
   **Remark:** This shows that the principal open subsets of $\operatorname{Spec} A$ form a basis for a topology on $\operatorname{Spec} A$, which is called the *Zariski topology.*

2. Let $f : A \to B$ be a ring homomorphism. By Exercise 2 of List 2, the association $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$ defines a map $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$. Show that $\varphi^{-1}(U_{A,a}) = U_{B,f(a)}$ for every $a \in A$.
   **Remark:** This shows that the map $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is a continuous map.

3. Describe the spectrum of the following rings: a field $K$, the integers $\mathbb{Z}$, and their quotient $\mathbb{Z}/6\mathbb{Z}$. Describe the maps of spectra that are induced by the inclusion $\mathbb{Z} \to \mathbb{Q}$ and the surjection $\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$.

***Exercise 7** (Bonus exercise)**.** [1]
Study the map $\varphi : \operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$ of spectra that is induced by the inclusion $\mathbb{Z} \to \mathbb{Z}[i]$ of $\mathbb{Z}$ into the Gaussian integers.

1. Show that if $\varphi(\mathfrak{q}) = \langle p \rangle$ for a prime number $p \in \mathbb{Z}$, then $\mathfrak{q}$ is a maximal ideal of $\mathbb{Z}[i]$ that is generated by a single prime element of $\mathbb{Z}[i]$.

2. Show that the fibres $\varphi^{-1}(\mathfrak{p})$ have either one or two elements for each fibre, and show that both cases occur.

   *Remark:* If $\varphi^{-1}(\mathfrak{p})$ has two elements, then we say that $\mathfrak{p}$ *splits in the extension* $\mathbb{Z} \to \mathbb{Z}[i]$. *Hint:* A useful tool is the Euclidean norm $N(a + ib) = a^2 + b^2$, which has some convenient properties (e.g. it is multiplicative and its fibres are finite).

3. Show that if $\mathfrak{p} = \langle p \rangle$ for some prime number $p \in \mathbb{Z}$ and if $\varphi^{-1}(\mathfrak{p}) = \{\mathfrak{q}\}$ consists of only one prime ideal $\mathfrak{q}$, then $\mathbb{Z}[i]/\mathfrak{q}$ is a field with $p$ or $p^2$ elements. Show that both cases occur.

   *Remark:* If $\mathbb{Z}[i]/\mathfrak{q}$ has $p^2$ elements, then we say that $\mathfrak{p}$ *is inert in the extension* $\mathbb{Z} \to \mathbb{Z}[i]$.

4. Show that if $\mathfrak{p} = \langle p \rangle$ for some prime number $p \in \mathbb{Z}$ and if $\varphi^{-1}(\mathfrak{p}) = \{\mathfrak{q}\}$ such that $\mathbb{Z}[i]/\mathfrak{q} \simeq \mathbb{Z}/\mathfrak{p}$, then $\mathfrak{q}$ is generated by an element $q \in \mathbb{Z}[i]$ such that $q^2 = p$.

   *Remark:* In this case, we say that $\mathfrak{p}$ *ramifies in the extension* $\mathbb{Z} \to \mathbb{Z}[i]$.

[1] Starred exercises are hard problem for those of you that search for a challenge. To balance the amount of work required to solve these exercises, starred exercises they are worth twice as many points as normal exercises.