Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
List 5 To be sent to xia.xiao@impa.br until 17.4.2020

Exercise 1.
Let A be a ring and $S \subset A$ a multiplicative subset.

1. Show that $\frac{ta}{ts} = \frac{a}{s}$, $\frac{s}{t} \cdot \frac{t}{s} = \frac{1}{1}$ and $\frac{a}{s} + \frac{b}{s} = \frac{a+b}{s}$ for all $a, b \in A$ and $s, t \in S$.
2. Show that $\frac{a}{s} = \frac{a'}{s'}$ if and only if $sa' = s'a$, in case that A is an integral domain.
3. Show that $S^{-1}A = \{0\}$ if $0 \in S$.
4. Let $A = A_1 \times A_2$ and $h = (1, 0)$. Show that the association $\frac{(a,b)}{h^i} \mapsto (a, 0)$ defines a ring isomorphism $A[h^{-1}] \simeq A_1$.

Exercise 2.
Let A be a ring and S a multiplicative subset. Show the following assertions.

1. The localization map $A \to S^{-1}A$ is injective if and only if for every $a \in S$, the multiplication $m_a : A \to A$ by a is an injective map.
2. If A is an integral domain, a unique factorization domain, a principal ideal domain or a field and $0 \notin S$, then $S^{-1}A$ is so, too.

Exercise 3.
Let A be a ring.

1. Show that $A[T_1, T_2] \simeq \big(A[T_1]\big)[T_2]$.
2. Let $h \in A$. Show that $A[h^{-1}] \simeq A[T]/\langle hT - 1\rangle$.

Exercise 4.
Let A be a ring, S a multiplicative subset of A and $\iota_S : A \to S^{-1}A$ the localization map. Show the following.

1. Given an ideal I of A, show that the ideal of $S^{-1}A$ generated by $\iota_S(I)$ equals
$$I \cdot S^{-1}A \;=\; \Big\{ \tfrac{a}{s} \in S^{-1}A \,\Big|\, a \in I, s \in S \Big\},$$
and that $I \cdot S^{-1}A$ is prime if I is prime and does not intersect S.
2. Show that for every prime ideal $\mathfrak{q}$ of $S^{-1}A$, the inverse image $\iota_S^{-1}(\mathfrak{q})$ is a prime ideal of A that does not intersect S.
3. Show that this defines mutually inverse bijections
$$\begin{array}{ccc} \Big\{\text{prime ideals } \mathfrak{p} \text{ of } A \text{ with } \mathfrak{p} \cap S = \emptyset\Big\} & \longleftrightarrow & \Big\{\text{prime ideals of } S^{-1}A\Big\} \\ \mathfrak{p} & \longmapsto & \mathfrak{p} \cdot S^{-1}A \\ \iota_S^{-1}(\mathfrak{q}) & \longleftarrow & \mathfrak{q} \end{array}$$

Exercise 5 (Bonus exercise)**.**
Show that the localization of a Euclidean domain is a Euclidean domain (or trivial). Find an example of a local ring A and a multiplicative subset S with $0 \notin S$ such that $S^{-1}A$ is not local.

Exercise 6 (Bonus exercise)**.**
Let A be a ring. Recall Exercise 6 of List 4 the definition of $\operatorname{Spec} A$ as the set of prime ideals $\mathfrak{p}$ of A together with the topology that is generated by the open subsets $U_{A,h} = \{\mathfrak{p} | h \notin \mathfrak{p}\}$.
Given a multiplicative subset S of A, show that the localization map $\iota_S : A \to S^{-1}A$ defines an injection $\varphi : \operatorname{Spec} A[h^{-1}] \to \operatorname{Spec} A$ that satisfies $\varphi(U_{A[h^{-1}], \frac{a}{s}}) = U_{A,ah}$ for every $a \in A$ and $s = h^i$ with $i \geq 0$.

Remark: This shows that $\varphi : \operatorname{Spec}(A[h^{-1}]) \to \operatorname{Spec} A$ is an open topological embedding with image U_h.