
Exercise 1.
Let K be a field and $M = N = K^2$, considered as additive groups. Define a map $K[T] \times M \to M$ by

$$\Big(\sum a_i T^i\Big).(m,n) \quad = \quad \Big(\sum (a_i m)\,,\, \sum a_i n\Big)$$

and a map $K[T] \times N \to N$ by

$$\Big(\sum a_i T^i\Big).(m,n) \quad = \quad \Big(\sum (a_i m + i a_i n)\,,\, \sum a_i n\Big)$$

where $a_i, m, n \in k$. Show that M and N are $K[T]$-modules with respect to these maps. Show that neither M nor N is simple, but that N is indecomposable while M is not.
Hint: T acts on M as the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and it acts on N as the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Exercise 2.

1. Let K be a field and $0 \to V_1 \to \cdots \to V_n \to 0$ an exact sequence of K-vector spaces. Show that $\sum (-1)^i \dim_K V_i = 0$.
2. Let A be a ring and $f : M \to N$ a homomorphism of A-modules that has a section $g : N \to M$, i.e. $f \circ g = \mathrm{id}_N$. Show that $M \simeq \ker f \oplus \mathrm{img}$.

Exercise 3 (Schur's lemma for algebras over algebraically closed fields)**.**
Let K be an algebraically closed field, A a K-algebra and V an irreducible A-module that is finite dimensional as a K-vector space. Show that every A-linear map $\phi : V \to V$ is of the form $\phi(v) = a.v$ for some $a \in K$.

Exercise 4.
Let K be a field and $M = N = k^2$ the $K[T]$-modules from Exercise 1. Let $P = K$.

1. Show that the map $K[T] \times P \to P$ with $\big(\sum a_i T^i\big).(m) = \sum a_i.m$ turns P into a $K[T]$-module.
2. Show that the inclusion $a \mapsto (a, 0)$ into the first coordinate defines injective $K[T]$-linear maps $i : P \to M$ and $j : P \to N$.
3. Show that there are short exact sequences of the form

   $$0 \longrightarrow P \xrightarrow{\ i\ } M \xrightarrow{\ p\ } P \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow P \xrightarrow{\ j\ } N \xrightarrow{\ q\ } P \longrightarrow 0$$

   for some $K[T]$-linear maps p and q.
4. Which of these sequences are split?

Exercise 5 (Bonus). Prove the *short 5-lemma*: given a ring A and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & M & \xrightarrow{p} & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_N} & & \downarrow{\scriptstyle f_M} & & \downarrow{\scriptstyle f_Q} & & \\ 0 & \longrightarrow & N' & \xrightarrow{i'} & M' & \xrightarrow{p'} & Q' & \longrightarrow & 0 \end{array}$$

of A-modules with exact rows, show that

1. f_M is a monomorphism if f_N and f_Q are monomorphisms,
2. f_M is an epimorphism if f_N and f_Q are epimorphisms, and
3. f_M is an isomorphism if f_N and f_Q are isomorphisms.