Part II: Representation theory of finite groups

1 Basic definitions and results

1.1 Representations

G finite group of order $n = \#G$

K field

Def: • A (finite dimensional K-linear) representation of G is a finite dimensional K-vector space V together with a group homomorphism

$$\rho : G \to GL(V).$$

→ Depending on the context, we write (ρ, V), ρ or V for a representation of G.

Rem: ρ defines the K-linear group action $G \curvearrowright V$, which is a map

$$\vartheta : G \times V \to V, \quad (g, v) \mapsto g.v = \rho(g).v$$

that satisfies

(1) $e.v = v$

(2) $(gh).v = g.(h.v)$

(3) $g.(v+w) = g.v + g.w$

(4) $g.(\lambda v) = \lambda(g.v)$

for all $g, h \in G$, $\lambda \in K$, $v, w \in V$.

Exercise: The association

$$\left\{ \begin{array}{c} \text{representations} \\ \rho : G \to GL(V) \end{array} \right\} \to \left\{ \begin{array}{c} K\text{-lin. gr. actions} \\ \vartheta : G \times V \to V \end{array} \right\}$$

$$\rho \mapsto \vartheta$$

is a bijection

Def: (ρ, V) repr. of G

- The dimension of (ρ, V) is $\dim \rho = \dim_K V$.
- It is faithful if ρ is injective.
- A subrepresentation of ρ is a subvector space $W \subset V$ that is stable (or invariant) under the action of G, i.e. $g.w \in W$ for all $g \in G$ and $w \in W$.
- ρ is irreducible if $V \neq \{0\}$ and if it has no subrepresentation other than $\{0\}$ and V.

Rem: If $W \subset V$ is a subrepr. of ρ, then ρ "restricts to W", i.e. $\exists!\ \rho|_W : G \to GL(W)$ s.t.

$\rho|_W(g).w = \rho(g).w$ for all $g \in G$, $w \in W$.

Def: (ρ, V) repr. of G

Fix an isomorphism $V \simeq K^d$ where $d = \dim V$, which yields an isomorphism $\alpha: GL(V) \xrightarrow{\sim} GL_d(K)$ of groups.

For $g \in G$, we call $A(g) = \alpha(\rho(g))$ a representation matrix of g; its coefficients $a_{ij}(g)$ $(i,j = 1, \dots, d)$ are called matrix coefficients of g.

Rem: Let $(e_1, \dots, e_d)$ be the canonical basis of K^d. Then the $a_{ij}(g)$ are determined by the formula

$$g.e_j = \sum_{k=1}^{d} a_{kj}(g).e_k$$

<u>Examples:</u> (0) <u>The trivial (d-dimensional) representation</u>

$$V \cong K^d$$

$$\rho_0 : G \longrightarrow GL(V) \cong GL_d(K)$$
$$g \longmapsto \mathrm{id}_V \quad \rightsquigarrow A(g) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

<u>note:</u> ρ_0 is irreducible iff. $\dim V = 1$.

(1) <u>1-dimensional representations</u>

$V = K$, $n = \# G$ (independent from the choice of $V \cong K$)

$$\rho : G \longrightarrow GL(V) = K^\times$$

- Since $\mathrm{ord}(g) \mid n$ for all $g \in G$ (Lagrange's thm.), $\rho(G) \subset \mu_n(K)$.
- The representation matrix $A(g) = (a_{11}(g))$ is independent of the choice of $V \cong K$, and $a_{11}(g) \in \mu_n(K)$.

$\rightarrow$ An 1-dim. real representation $\rho : G \to \mathbb{R}^\times$ corresponds to a choice of subgroup $H = \ker \rho < G$ of index 1 or 2 since $\mu_n(\mathbb{R}) \subset \{\pm 1\}$ for all $n \geq 1$.

$\rightarrow$ There are n complex representations of $\mathbb{Z}/n\mathbb{Z}$, which are

$$\rho_i : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^\times \qquad (i = 1, \ldots, n)$$
$$[1] \longmapsto \zeta_n^i$$

$\rightarrow$ $\rho : G \to K^\times$ (any G, any K) <u>Fact:</u> $[G,G] \trianglelefteq G$

$[G,G] = \langle aba^{-1}b^{-1} \mid a,b \in G \rangle$ (<u>commutator subgroup of G</u>)

$G^{ab} = G/[G,G]$ ("largest" abelian quotient)

Then $[G,G] \subset \ker \rho$ and ρ factors uniquely into

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "g \mapsto \bar{g}"'] & K^\times \\ G^{ab} \arrow[ur, dashed, "\rho^{ab}"'] & \end{tikzcd}$$

(Exercise!)

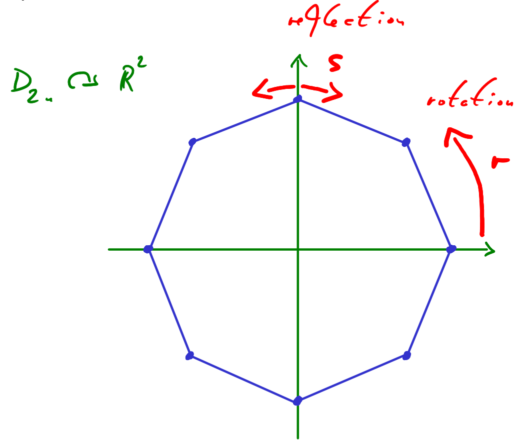
<u>Consequence:</u> If G is simple and not abelian, then every 1-dim. repr. of G is trivial.

(2) <u>2-dimensional representations</u>

$\rightarrow$ $D_n = \langle r, s \mid r^n = s^2 = (rs)^2 = e \rangle$

(<u>dihedral group</u>)

has the real representation



symmetry group of the regular n-gon

$$\begin{aligned} \rho: D_n &\longrightarrow GL_2(\mathbb{R}) \\ r &\longmapsto \begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix} \\ s &\longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This representation is irreducible for $n \geq 3$.

(3) <u>Permutation representations</u>

S set

$\vartheta: G \times S \longrightarrow S$
$(g, x) \longmapsto g.x$

<u>G-action</u>, i.e.
(1) $e.x = x$
(2) $(gh).x = g.(h.x)$
for all $g, h \in G$, $x \in S$

$K^S = \mathrm{Fun}(S, K) = \{ f: S \to K \}$

is a K-v.s. with basis $\left\{ \begin{aligned} \delta_x : S &\longrightarrow K \\ y &\longmapsto \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \end{aligned} \right\}_{x \in S}$

ϑ defines the <u>permutation representation</u>

$$\rho: G \longrightarrow GL(K^S) \simeq GL_S(K) \quad (\text{w.r.t. } \{\delta_x\}_{x\in S})$$

$$g \longmapsto \rho(g) = \begin{pmatrix} \text{permutation matrix } A(g) = (a_{xy}) \\ \text{with } a_{xy} = \begin{cases} 1 & \text{if } x = g.y \\ 0 & \text{if } x \neq g.y \end{cases} \end{pmatrix}$$

with $\quad g.f : x \longmapsto f(\underbrace{g^{-1}.x}_{=\vartheta(g^{-1},x)})$.

<u>note:</u> • $\dim \rho = \# S$

• $(gh).f(x) = f((gh)^{-1}.x) = f((h^{-1}g^{-1}).x) = f(h^{-1}.(g^{-1}.x))$

$= h.f(g^{-1}.x) = g.(h.f(x))$

• in general, ρ is not irreducible

$\rightarrow$ If $S = G$ and $\vartheta: G \times G \longrightarrow G$, then
$(g, h) \longmapsto gh$

$$\rho: G \longrightarrow GL(K^G)$$

is called the <u>regular representation of G.</u>

<u>note:</u> • $\dim \rho = \# G$

• <u>later:</u> every irreducible representation of G occurs as a subrepresentation of ρ.

(4) <u>Galois representations</u>

L/K finite Galois extension

$G = \mathrm{Gal}(L/K)$

Then $\rho: G \longrightarrow GL(L)$ is a K-linear representation.
$\sigma \longmapsto [\sigma : L \to L]$

<u>Later:</u> ρ is (isomorphic to) the regular representation of G.

<u>Consequence:</u> Normal basis theorem.

(5) $\rho : G \longrightarrow GL(V)$ any representation

$V^G = \{ v \in V \mid g.v = v \text{ for all } g \in G \}$ (fixed vectors of ρ)

Then $V^G \subset V$ is the largest trivial subrepresentation.