1.2 Equivariant homomorphisms

G finite group

K field

Def: V, W K-linear G-repr.

- A <u>G-equivariant homomorphism</u> ("<u>G-hom</u>") is a K-linear map $f: V \to W$ such that $f(g.v) = g.f(v)$ for all $g \in G$, $v \in V$,

  i.e.
$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "g_V(g)"'] & W \arrow[d, "g_W(g)"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$
  commutes for all $g \in G$.

- $Rep_K(G)$ is the category of K-linear G-rep's and G-hom's, and $Hom_G(V, W) = \{G\text{-hom's } f: V \to W\}$.

- V and W are <u>isomorphic</u> if there is a bijective G-hom. $f: V \to W$.

(<u>Exercise:</u> $f: V \to W$ bijective G-hom. $\Rightarrow$ inverse bijection $g: W \to V$ is also a G-hom.)

Ex: (0) Any two G-representations V and W admit the <u>trivial</u> G-hom. $0: V \to W$, $v \mapsto 0$.

(1) $id_V: V \to V$, $v \mapsto v$ is a G-hom.

(2) More generally, if $W \subset V$ is a subrepr., then the inclusion map $W \hookrightarrow V$ is a G-hom.

(3) $f, g : V \to W$ G-hom's, $\lambda \in K$

Then $f+g : V \to W$, $v \mapsto f(v)+g(v)$ and $\lambda \cdot f : V \to W$, $v \mapsto \lambda f(v)$ are G-hom's. (exercise!)

(4) $\mathrm{Hom}_K(V, W) = \{K\text{-linear maps } f : V \to W\}$

is a G-repr. via

$$g.f : v \longmapsto g.\left(f(g^{-1}.v)\right)$$

$$\begin{tikzcd} V \arrow[r, "f"] & W \arrow[d, "g"] \\ V \arrow[u, "g^{-1}"] \arrow[r, "gf"] & W \end{tikzcd}$$

We have $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_K(V, W)^G$.

Lemma 1: $W \subset V$ subrepr.

Then the quotient vector space V/W is a G-repr. via $g.\bar{v} = \overline{g.v}$, and the projection $\pi : V \to V/W$, $v \mapsto \bar{v}$ is a G-hom.

proof: • $G \curvearrowright V/W$ well-defined: $\overline{v'} = \bar{v}$, i.e. $v' = v + w$ for some $w \in W$.

$$\Rightarrow g.\overline{v'} = \overline{g.(v+w)} = \overline{g.v + g.w} = \overline{g.v} + \underbrace{\overline{g.w}}_{=0} = \overline{g.v} = g.\bar{v}.$$

• all other properties of a G-repr. are inherited from $G \curvearrowright V$:

- $e.\bar{v} = \overline{e.v} = \bar{v}$;
- $(gh).\bar{v} = \overline{(gh).v} = \overline{g.(h.v)} = g.(\overline{h.v}) = g.(h.\bar{v})$;
- $g.(\bar{v} + \bar{w}) = \overline{g.(v+w)} = \overline{g.w} + \overline{g.v} = g.\bar{v} + g.\bar{w}$;
- $g.(\lambda \bar{v}) = \overline{g.(\lambda v)} = \overline{\lambda (g.v)} = \lambda.(g.\bar{v})$.

• By def, $\pi(g.v) = \overline{g.v} = g.\bar{v} = g.\pi(v) \Rightarrow \pi$ is a G-hom. $\square$

Lemma 2: $f: V \to W$ G-hom.

Then (1) $\ker f = \{ v \in V \mid f(v) = 0 \}$ is a subrepr. of V;

(2) $\operatorname{im} f = \{ f(v) \mid v \in V \}$ is a subrepr. of W.

proof: (1) $f(g.v) = g.f(v) = 0$ for all $g \in G$, $v \in \ker f$

$\Rightarrow g.v \in \ker f \Rightarrow \ker f$ is G-invariant.

(2) $g.f(v) = f(g.v) \in \operatorname{im} f$ for all $g \in G$, $v \in V$

$\Rightarrow \operatorname{im} f$ is G-invariant. □

Def: V G-repr.

A homothety of V (with center 0) is a G-hom.

of the form $\lambda \cdot \mathrm{id}_V : V \to V$.
$v \mapsto \lambda . v$

Lemma 3 (Schur's Lemma)

V, W irred. repr. of G

$f: V \to W$ G-hom.

Then (1) f is 0 or an isomorphism.

(2) If $V = W$ and K is algebraically closed,

then f is a homothety.

proof: (1) V irred. $\Rightarrow \ker f = \{0\}$ or V

$\Rightarrow f$ inj. or 0

If f is inj., then $\operatorname{im} f \neq \{0\}$. Since W is irred.,

$\operatorname{im} f = W$. Thus f is an isom. if not 0. ↯

(2) If K is alg. cl., $\varphi : V \to V$ has an eigenvector $v_0 \neq 0$ with eigenvalue λ. Define

$$\varphi_\lambda = \varphi - \lambda \cdot id_V : V \to V,$$

which is a G-hom. with

$$\varphi_\lambda(v_0) = \varphi(v_0) - \lambda \cdot v_0 = 0,$$

i.e. $v_0 \in \ker \varphi_\lambda$. Since V is irred., $\ker \varphi_\lambda = V$.

Thus $\varphi_\lambda(v) = 0$ ($\Leftrightarrow \varphi(v) = \lambda \cdot v$) for all $v \in V$

$\Rightarrow \varphi = \lambda \cdot id_V$. $\square$

<u>Thm. 4</u> (Maschke's Theorem)

$W \subset V$ subrepr.

Assume that $\operatorname{char} K \nmid n = \# G$.

Then there is a G-hom. $\pi : V \to W$ such that $\pi|_W = id_W$.

$$\begin{tikzcd} W \arrow[r, "\iota"] \arrow[dr, "id_W"'] & V \arrow[d, "\pi"] \\ & W \end{tikzcd}$$

proof: • Let $\pi_0 : V \to W$ be a K-linear surjection such that $\pi_0(w) = w$ for all $w \in W$. Define

$$\pi(v) = \frac{1}{n} \sum_{g \in G} \underbrace{g . \big(\underbrace{\pi_0 (g^{-1} . v)}_{\in \operatorname{im} \pi_0 = W} \big)}_{\in g . W = W} \quad \in W,$$

which yields a K-linear map $\pi : V \to W$.

• π is G-equivariant since

$$\pi(g.v) = \frac{1}{n} \sum_{h \in G} h . \pi_0 (h^{-1} . (g . v))$$

$$= \frac{1}{n} \sum_{h\in G} g.\left((g^{-1}h).\pi_0\left((g^{-1}h)^{-1}.v\right)\right)$$

$$\underset{(g^{-1}h \mapsto h)}{=} g.\left(\frac{1}{n}\sum_{h\in G} h.\pi_0(h^{-1}.v)\right) = g.\pi(v).$$

- For $w \in W$,

$$\pi(w) = \frac{1}{n}\sum_{g\in G} g.\pi_0(\underbrace{g^{-1}.w}_{\in W})$$

$$= \frac{1}{n}\sum_{g\in G} \underbrace{g.(g^{-1}.w)}_{=w} = \frac{1}{n}.\underbrace{(\# G)}_{=n}.w = w. \qquad \square$$

Rem: An immediate consequence of Maschke's Theorem is that every subrepresentation $W \subset V$ has a complement, i.e. $V = W \oplus \ker \pi$ (as K-vector spaces)

Later: Every representation decomposes (uniquely) into a direct sum of irred. representations.

Rem: The hypothesis char $K \nmid \# G$ cannot be omitted as the following example shows:

$G = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) = \{id, \mathrm{Frob}_2\} \simeq \mathbb{Z}/2\mathbb{Z}$ acts $\mathbb{F}_2$-linearly on $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ (where $\alpha^2+\alpha+1=0$).

Then $\mathbb{F}_2 \subset \mathbb{F}_4$ is G-invariant, but neither $\{0, \alpha\}$ nor $\{0, \alpha^2\}$ is ($\mathrm{Frob}_2(\alpha) = \alpha^2$).

Thus $\mathbb{F}_2 \subset \mathbb{F}_4$ has no G-invariant complement.