1.4 Categorical properties (continued)

G finite group
K field

Def: $V_1 \dots V_r$ G-repr.

The <u>tensor product of $V_1 \dots V_r$</u> is the K-vector space

$$\bigotimes_{i=1}^{r} V_i = V_1 \otimes_K \dots \otimes_K V_r$$

together with the G-action $G \curvearrowright \bigotimes V_i$ defined by

$$g.\left(\sum_i a_i \cdot (v_{i,1} \otimes \dots \otimes v_{i,r})\right) = \sum a_i \cdot (g.v_{i,1} \otimes \dots \otimes g.v_{i,r}).$$

<u>Rem:</u> $\bigotimes V_i$ is indeed a G-repr.:

- <u>well-def:</u> $g.(v_1 \otimes \dots \otimes (av_k + bw_k) \otimes \dots \otimes v_r)$

$$= g.v_1 \otimes \dots \otimes g.(av_k + bw_k) \otimes \dots \otimes g.v_r$$

$$= g.v_1 \otimes \dots \otimes (ag.v_k + bg.w_k) \otimes \dots \otimes g.v_r$$

$$= a \cdot (g.v_1 \otimes \dots \otimes g.v_k \otimes \dots \otimes g.v_r) + b \cdot (g.v_1 \otimes \dots \otimes g.w_k \otimes \dots \otimes g.v_r)$$

$$= ag.(v_1 \otimes \dots \otimes v_k \otimes \dots \otimes v_r) + bg.(v_1 \otimes \dots \otimes w_k \otimes \dots \otimes v_r)$$

<u>note:</u> K-linearity (axioms 3 & 4) is similarly verified.

(1) $e.(v_1 \otimes \dots \otimes v_r) = e.v_1 \otimes \dots \otimes e.v_r = v_1 \otimes \dots \otimes v_r$

$g.(0 \otimes \dots \otimes 0) = g.0 \otimes \dots \otimes g.0 = 0 \otimes \dots \otimes 0$

(2) $(gh).(v_1 \otimes \dots \otimes v_r) = ((gh).v_1 \otimes \dots \otimes (gh).v_r)$
$= (g.(h.v_1) \otimes \dots \otimes g.(h.v_r))$
$= g.(h.(v_1 \otimes \dots \otimes v_r))$

<u>Def:</u> $V_1 \dots V_r, W$ G-repr.

A map
$$f: V_1 \oplus \dots \oplus V_r \longrightarrow W$$
is a <u>K-multilinear G-equivariant map</u> if

(1) $f(v_1 \dots v_{k-1}, av_k + bw_k, v_{k+1} \dots v_r)$ (K-multilinear)

$= a f(v_1 \dots v_k \dots v_r) + b f(v_1 \dots w_k \dots v_r)$

(2) $f(g.(v_1 \dots v_r)) = g. f(v_1 \dots v_r)$ (G-equivariant)

for all $k \in \{1 \dots r\}$, $v_i \in V_i$ $(i = 1 \dots r)$, $w_k \in V_k$, $a, b \in K$, $g \in G$.

<u>Rem:</u> $p: V_1 \oplus \dots \oplus V_r \longrightarrow V_1 \otimes_K \dots \otimes_K V_r$

$(v_1 \dots v_r) \longmapsto v_1 \otimes \dots \otimes v_r$

is K-multilinear G-equiv.

<u>Prop 5:</u> $V_1 \dots V_r$ G-repr.

Then $\bigotimes V_i$ together with $p: \bigoplus V_i \to \bigotimes V_i$

satisfies the following universal property:

for every K-multilin. G-equiv. map $f: \bigoplus V_i \to W$,

there exists a unique G-hom. $\bar{f}: \bigotimes V_i \to W$

s.t.
$$\begin{tikzcd} \bigoplus V_i \arrow[r, "f"] \arrow[d, "p"'] & W \\ \bigotimes V_i \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$
commutes.

proof: • Define $\bar{\varphi}\left(\sum_i a_i (v_{i,1} \otimes \dots \otimes v_{i,r})\right) = \sum_i a_i \varphi(v_{i,1}, \dots, v_{i,r})$.

• <u>well-def as a K-linear map:</u>

$$\bar{\varphi}(v_1 \otimes \dots \otimes (a v_k + b w_k) \otimes \dots \otimes v_r)$$
$$= \varphi(v_1, \dots, a v_k + b w_k, \dots, v_r)$$
$$= a \varphi(v_1, \dots, v_k, \dots, v_r) + b \varphi(v_1, \dots, w_k, \dots, v_r)$$
$$= a \bar{\varphi}(v_1 \otimes \dots \otimes v_k \otimes \dots \otimes v_r) + b \bar{\varphi}(v_1 \otimes \dots \otimes w_k \otimes \dots \otimes v_r)$$

• By def., we have $\varphi = \bar{\varphi} \circ p$. Since $\bigotimes V_i$ is generated by "pure tensors" $v_1 \otimes \dots \otimes v_r$, $\bar{\varphi}$ is uniquely determined by

$$\bar{\varphi}(v_1 \otimes \dots \otimes v_r) = (\bar{\varphi} \circ p)(v_1, \dots, v_r) = \varphi(v_1, \dots, v_r).$$

• <u>G-equiv.:</u>

$$\bar{\varphi}(g.(v_1 \otimes \dots \otimes v_r)) = \bar{\varphi}(g.v_1 \otimes \dots \otimes g.v_r)$$
$$= \varphi(g.v_1, \dots, g.v_r)$$
$$= g.\varphi(v_1, \dots, v_r)$$
$$= g.\bar{\varphi}(v_1 \otimes \dots \otimes v_r).$$

$$\begin{tikzcd} V \arrow[r, "\varphi"] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[u, "\rho_V(g^{-1})"] \arrow[r, "g.\varphi"] & W \end{tikzcd}$$

$\square$

<u>recall:</u> V, W G-repr.

Then $\mathrm{Hom}_K(V, W)$ is a G-repr. w.r.t.

$$(g.\varphi)(v) = g.\varphi(g^{-1}.v) = \rho_W(g).\left(\varphi(\rho_V(g^{-1}).v)\right)$$

<u>Def:</u> V G-repr.
K trivial 1-dim G-repr.

The <u>contragredient representation of V</u> is

$$V^* = \mathrm{Hom}_K(V, K).$$

Lemma 6: V G-repr.

$(e_1, \dots, e_d)$ K-basis of V $\left(\text{i.e. } g.e_j = \sum_{k=1}^{d} a_{k,j}(g).e_k \right)$

$g \in G$

$A(g) = (a_{ij}(g))_{i,j}$ representation matrix of g (w.r.t. $(e_1, \dots, e_d)$)

V^* contragr. repr. of V $\left(\text{i.e. } e_j^* : V \to K,\ e_i \mapsto \delta_{ij} \right)$

$(e_1^*, \dots, e_d^*)$ dual basis of $(e_1, \dots, e_d)$

Then the representation matrix of g w.r.t. $(e_1^*, \dots, e_d^*)$

is $A^*(g) = A(g^{-1})^T = (a_{ji}(g^{-1}))_{i,j=1,\dots,d}$.

proof: Let $A^*(g) = (a_{ij}^*(g))$. By def. of $A^*(g)$,

$$g.e_j^* = \sum_{k=1}^{d} a_{kj}^*(g)\, e_k^*$$

$$\Rightarrow g.e_j^*(e_i) = \sum_{k=1}^{d} a_{k,j}^*(g)\, \underbrace{e_k^*(e_i)}_{=\delta_{k,i}} = a_{ij}^*(g).$$

- On the other hand,

$$(g.e_j^*)(e_i) = g.\left(e_j^*(g^{-1}.e_i)\right) \underset{\uparrow}{=} e_j^*(g^{-1}.e_i)$$

(K is a trivial G-repr.)

$$= e_j^*\left(\sum_{k=1}^{d} a_{k,i}(g^{-1})\, e_k\right)$$

$$= \sum_{k=1}^{d} a_{k,i}(g^{-1}) \cdot \underbrace{e_j^*(e_k)}_{=\delta_{j,k}} = a_{j,i}(g^{-1})$$

Thus $a_{ij}^*(g) = a_{ji}(g^{-1})$.

$\square$

Cor 7: V G-repr.

Then $(V^*)^* \simeq V$.

proof: Choose a basis $(e_1, \ldots, e_d)$ for V. Then

$$A^{**}(g) = (A^*(g^{-1}))^T = \left(\left(A((g^{-1})^{-1})\right)^T\right)^T = A(g)$$

for all $g \in G$. $\square$

Ex: $G = \langle g \rangle$ cyclic of order n

$\zeta_n \in \mathbb{C}$ prim. n-th root of 1

$$\begin{aligned} \rho_k : G &\longrightarrow \mathbb{C}^\times = GL_1(\mathbb{C}) \\ g^i &\longmapsto \zeta_n^{ki} \end{aligned}$$

Then w.r.t. any basis of $\mathbb{C}$,

$$A(g^i) = (\zeta_n^{ki}) \qquad (\text{as a } (1\times 1)\text{-matrix})$$

Thus $A^*(g^i) = A(g^{-i})^T = (\zeta_n^{-ki})$

$\Rightarrow \rho_k^* \simeq \rho_{n-k}$.