2.3 Examples

Fact: $\phi: G \to H$ gr. hom

The inverse image $\phi^{-1}(c)$ of a conjugacy class $c \subset H$ is a union of conjugacy classes in G.

(1) $D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = e \rangle$

$$\begin{tikzcd} D_4 \arrow[r, "2:1"] \arrow[drr, bend right, "\text{1-dim. characters}"'] & D_4/\langle r^2\rangle = \langle r,s \mid r^2 = s^2 = (rs)^2 = e\rangle = D_4^{ab} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[d, "4 \text{ gr. hom.}"] \\ & \mathbb{C}^\times \end{tikzcd}$$

$r, s \mapsto \pm 1$

[thus: $rs = (rs)^{-1} = s^{-1}r^{-1} = sr \Rightarrow$ abelian]

conj. cl.	e	r^2	r, r^{-1}	s, r^2s	rs, r^3s
# elts.	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

note: • conj. cl. of $D_4/\langle r^2\rangle$ are singletons $\{\bar e\}, \{\bar r\}, \{\bar s\}, \{\overline{rs}\}$

$\Rightarrow \# c \le 2$ for conj. cl. $c \subset D_4$

• $srs^{-1} = r^{-1}$

$rsr^{-1} = r^2 s$

$r(rs)r^{-1} = r^3 s$

• Since $\# D_4 = 8$ and $8 - 4 \cdot 1^2 = 4 = 2^2$,

$\dim \chi_5 = 2$ and $\chi_5 = \frac{1}{2}\left(\chi_{reg} - \sum_{i=1}^{4} \chi_i\right)$.

(2) $Q_8 = \langle i, j, k \mid ij = k,\ i^2 = j^2 = k^2,\ i^4 = e \rangle$ (quaternion group)

$= \{\pm 1, \pm i, \pm j, \pm k\}$ for $-1 = i^2 = j^2 = k^2$, $-i = (-1)\cdot i = i^3, \ldots$

note: non-abelian, $Z(Q_8) = \{\pm 1\}$

$$\begin{tikzcd}
Q_8 \arrow[r, "2:1", "\pi"'] \arrow[dr, "\text{1-dim. characters}"'] & Q_8/\{\pm 1\} = Q_8^{ab} = \{\bar{1}, \bar{i}, \bar{j}, \bar{k}\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[d, "\text{4 hom's}"] \\
 & \mathbb{C}^\times
\end{tikzcd}
\qquad
\begin{array}{c} \bar{i}, \bar{j} \\ \downarrow \\ \pm 1 \end{array}$$

conj. cl.	1	-1	$\pm i$	$\pm j$	$\pm k$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

note: $i, j, k \notin Z(Q_8)$
$\Rightarrow \# C(i) \geq 2$
$\Rightarrow \# C(i) = \pi^{-1}(\bar{i}) = \{\pm i\}$,
and similar for j, k.

As in the case of D_4, $\# Q_8 = 8$ and $8 - 4 \cdot 1^2 = 4 = 2^2$ imply $\dim \chi_5 = 2$ and $\chi_5 = \frac{1}{2}\left(\chi_{reg} - \sum_{i=1}^{4} \chi_i\right)$

Rem: D_4 and Q_8 have the same character table, but are not isomorphic:

$$\#\{g \in D_4 \mid \operatorname{ord} g = 2\} = 5 \neq 1 = \#\{g \in Q_8 \mid \operatorname{ord} g = 2\}$$

Fact: $\sigma, \tau \in S_n$

Then $C(\sigma) = C(\tau)$, i.e. $\sigma = \gamma \tau \gamma^{-1}$ for some $\gamma \in S_n$, iff. σ and τ have the same cycle type (= lengths l_i of the cycles in $\sigma = (i_{1,1} - i_{1,l_1}) \cdots (i_{r,1} - i_{r,l_r})$)

(3) $S_4 \xrightarrow{12:1} S_4^{ab} = S_4 / A_4 \simeq \mathbb{Z}/2$

$$\begin{tikzcd} S_4 \arrow[r, "12:1"] \arrow[dr, "\chi_{triv},\ \chi_{sign}"'] & S_4^{ab} = S_4/A_4 \simeq \mathbb{Z}/2 \arrow[d, "trivial / sign"] \\ & \mathbb{C}^\times \end{tikzcd}$$

conj. cl.	e	(12)	(123)	(1234)	$(12)(34)$
# elts.	1	6	8	6	3
$\chi_1 = \chi_{triv}$	1	1	1	1	1
$\chi_2 = \chi_{sign}$	1	-1	1	-1	1
$\chi_3 = \tilde{\chi} \circ \pi$	2	0	-1	0	2
$\chi_4 = \tilde{\chi}_4 - \chi_1$	3	1	0	-1	-1
$\chi_5 = \chi_4 \cdot \chi_2$	3	-1	0	1	-1

- Dimensions:

$24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$

- $\langle \chi_3, \chi_3 \rangle = \frac{1}{24}[1 \cdot 2^2 + 8 \cdot (-1)^2 + 3 \cdot 2^2]$
$= \frac{1}{24}[4 + 8 + 12] = 1$

- $\langle \chi_4, \chi_4 \rangle = \frac{1}{24}[1 \cdot 3^2 + 6 \cdot 1^2 + 6 \cdot (-1)^2 + 3 \cdot (-1)^2]$
$= \frac{1}{24}[9 + 6 + 6 + 3] = 1$

- $S_4 \xrightarrow[\pi]{4:1} S_4 / V \simeq S_3 \xrightarrow[\text{irred.}]{\tilde{\varsigma}_3} GL_2(\mathbb{C})$

$V = \{(ij)(kl) \mid \{1\_4\} = \{ijkl\}\} \cup \{e\}$

Klein four group

conj. cl.	e	(12)	(123)
#	1	3	2
$\tilde{\chi} = \chi_{\tilde{\varsigma}_3}$	2	0	-1

- $\tilde{\varsigma}_4 : S_4 \longrightarrow GL_4(\mathbb{C})$ permutation repr. from $S_4 \circlearrowright \{1\_4\}$

recall: $\tilde{\chi}_4(g) = \#\{\text{fixed pts. of } g \circlearrowright \{1\_4\}\}$ for $\tilde{\chi}_4 = \chi_{\tilde{\varsigma}_4}$

conj. cl.	e	(12)	(123)	(1234)	$(12)(34)$
# elts.	1	6	8	6	3
$\tilde{\chi}_4$	4	2	1	0	0
$\chi_1 = \chi_{triv}$	1	1	1	1	1
$\chi_4 = \tilde{\chi}_4 - \chi_1$	3	1	0	-1	-1

- $\langle \tilde{\chi}_4, \chi_1 \rangle = \frac{1}{24}[1 \cdot 1 \cdot 4 + 6 \cdot 1 \cdot 2 + 8 \cdot 1 \cdot 1]$
$= \frac{1}{24}[4 + 12 + 8] = 1$

- $\langle \chi_4, \chi_4 \rangle = \frac{1}{24}[1 \cdot 3^2 + 6 \cdot 1^2 + 6 \cdot (-1)^2 + 3 \cdot (-1)^2]$
$= \frac{1}{24}[9 + 6 + 6 + 3] = 1$

Compute tensor products

(4) S_3

conj. cl.	e	(12)	(123)
# elts.	1	3	2
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
$\chi_2^2 = \chi_1$	1	1	1
$\chi_2\chi_3 = \chi_3$	2	0	-1
$\chi_3^2 = \chi_1 + \chi_2 + \chi_3$	4	0	1

$\otimes$	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_2	V_1	V_3
V_3	V_3	V_3	$V_1 \oplus V_2 \oplus V_3$

$$\langle \chi_3^2, \chi_1 \rangle = \frac{1}{6}[1 \cdot 4 \cdot 1 + 2 \cdot 1 \cdot 1] = 1$$

$$\langle \chi_3^2, \chi_2 \rangle = \frac{1}{6}[1 \cdot 4 \cdot 1 + 2 \cdot 1 \cdot 1] = 1$$

$$\langle \chi_3^2, \chi_3 \rangle = \frac{1}{6}[1 \cdot 4 \cdot 2 + 2 \cdot 1 \cdot (-1)] = 1$$

(5) $G = A_4$ (use Exercise 3 from List 13)

conj. cl.	e	$(12)(34)$	(123)	(132)
# elts	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	ζ_3	ζ_3^2
χ_3	1	1	ζ_3^2	ζ_3
χ_4	3	-1	0	0
χ_4^2	9	1	0	0

for $i = 1, 2, 3$,

$$\langle \chi_4^2, \chi_i \rangle = \frac{1}{12}[1 \cdot 9 \cdot 1 + 3 \cdot 1 \cdot 1] = 1$$

$$\langle \chi_4^2, \chi_4 \rangle = \frac{1}{12}[1 \cdot 9 \cdot 3 + 3 \cdot 1 \cdot (-1)] = 2$$

Thus $\chi_4^2 = \chi_1 + \chi_2 + \chi_3 + 2\chi_4$

and $V_4 \otimes V_4 \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4^2$

Concluding remark

(1) For $K = \bar{K}$ of char. $p \nmid \# G$, we can use Brauer characters:

- fix an isomorphism $\mu_\varepsilon(K) \xrightarrow{\sim} \mu_\varepsilon(\mathbb{C})$, $\lambda \longmapsto \hat{\lambda}$ where $\varepsilon = \exp(G)$;
- the Brauer character of a G-repr. ρ of dim. d over K is

$$\hat{\chi} : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \sum_{i=1}^{d} \hat{\lambda}_i \qquad \text{where } \rho(g) \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$$

(2) If K is not algebraically closed, but $\# \mu_\varepsilon(K) = \varepsilon$, we can use:

Thm: $\rho : G \to GL(V)$ irred. G-repr. over K

$\bar{V} = V \otimes_K \bar{K}$

$\bar{\rho} : G \xrightarrow{\rho} GL(V) \hookrightarrow GL(\bar{V})$ G-repr. over $\bar{K}$

Then $\bar{\rho}$ is irreducible.

(not so easy to prove...)

Cor: V irred. G-repr. over K

Then $\mathrm{Hom}_G(V, V) = K$.