Exercises for Algebra 2
List 12 To hand in at 9.11.2020

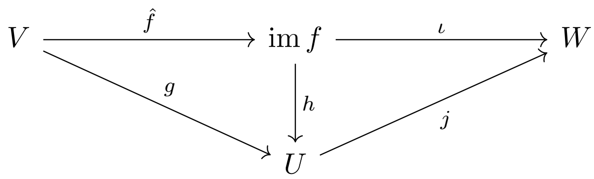
Exercise 1.
Let $f : V \to W$ be a morphism in $\mathrm{Rep}_K(G)$, i.e. a G-equivariant homomorphism. Show that

1. f is a monomorphism if and only if f is injective;
2. f is an epimorphism if and only if f is surjective;
3. f is an isomorphism if and only if f is bijective.

Show that every monomorphism in $\mathrm{Rep}_K(G)$ is a kernel and that every epimorphism in $\mathrm{Rep}_K(G)$ is a cokernel.

Exercise 2.
Let $f : V \to W$ be a morphism in $\mathrm{Rep}_K(G)$ and $\operatorname{im} f$ its image, which comes together with the restriction $\hat{f} : V \to \operatorname{im} f$ of f to $\operatorname{im} f$ and the inclusion $\iota : \operatorname{im} f \to W$ as a subrepresentation.

1. Show that $\operatorname{im} f$ together with the restriction $\hat{f} : V \to \operatorname{im} f$ of f to $\operatorname{im} f$ and the inclusion $\iota : \operatorname{im} f \to W$ is the categorical image of f, i.e. ι is a monomorphism and for every other representation U, morphism $g : V \to U$ and monomorphism $j : U \to W$ such that $f = j \circ g$, there is a unique morphism $h : \operatorname{im} f \to U$ such that the diagram



commutes.

2. Let $\mathcal{C}$ be a category with zero object such that every morphism has a kernel and a cokernel. Let $f : V \to W$ be a morphism and $\pi : W \to \operatorname{coker} f$ the canonical projection to the cokernel of f. Show that the canonical inclusion $\iota : \ker \pi \to W$ is a monomorphism and that there is a unique morphism $\hat{f} : V \to \ker \pi$ such that $f = \iota \circ \hat{f}$. Show that $\ker \pi$ together with $\hat{f}$ and ι is a categorical image of f.

Exercise 3.
Let $D_3 = \langle r, s | r^3 = s^2 = (rs)^2 = e \rangle$ act on the $V = \mathbb{R}^2$ as the symmetry group of the regular triangle (with center $(0,0)$ and a corner in $(1,0)$). Describe matrix representations of r and s for V and V^* w.r.t. to the standard basis of $\mathbb{R}^2$ (and its dual). Describe the matrix representations of r and s for $V \otimes V^*$. Show that $V \otimes V^*$ is irreducible.

Remark: We will later see that the complexification of $V \otimes V^*$ is not irreducible (as a complex representation).

Exercise 4.
Let U, V and W be representations of G over a field K. Find an isomorphism

$$\mathrm{Hom}_K(U \otimes_K V, W) \longrightarrow \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W))$$

of G-representations. Conclude that $\mathrm{Hom}_K(U, V) \simeq U^* \otimes_K V$ as G-representations. Conclude that if $V = U$ is 1-dimensional, then $V^* \otimes_K V$ is isomorphic to the trivial 1-dimensional representation of G over K.

Bonus: Show that $- \otimes_K V$ and $\mathrm{Hom}_K(V, -)$ are naturally functors from $\mathrm{Rep}_K(G)$ to $\mathrm{Rep}_K(G)$, and that $- \otimes_K V$ is left-adjoint to $\mathrm{Hom}_K(V, -)$.

Exercise 5 (Bonus)**.**
Let G be a finite group. Let X be the set of isomorphism classes $[V]$ of complex representations V of G.

1. Show that $([V_1], [W_1]) \sim ([V_2], [W_2])$ if and only if $V_1 \oplus W_2 \simeq V_2 \oplus W_1$ defines an equivalence relation $\sim$ on $X \times X$. Define the *(0-th) K-group of G* as the quotient set
$$K_0(G) \;=\; X \times X/\sim .$$
We write $V - W$ for the equivalence class of $([V], [W])$ in $K_0(G)$ and call $V - W$ a *virtual representation of G*. We write $[V]$ for $V - \{0\}$ where $\{0\}$ is the zero-dimensional representation.

2. Show that the addition
$$(V_1 - W_1) + (V_2 - W_2) \;=\; V_1 \oplus V_2 \;-\; W_1 \oplus W_2$$
and the multiplication
$$(V_1 - W_1) \cdot (V_2 - W_2) \;=\; (V_1 \otimes V_2) \oplus (W_1 \otimes W_2) \;-\; (V_1 \otimes W_2) \oplus (V_2 \otimes W_1)$$
turn $K_0(G)$ into a ring whose zero is $[\{0\}]$ and whose one is $[\mathbb{C}]$ where $\mathbb{C}$ is the trivial one-dimensional representation.

3. Show that $K_0(G)$ is generated over $\mathbb{Z}$ by the classes $[V_1], \ldots, [V_s]$ of the irreducible representations of G, i.e. $K_0(G)$ is a quotient of $\mathbb{Z}[T_1, \ldots, T_s]$.

4. Show that the association $[V] \mapsto \dim_{\mathbb{C}} V$ extends to a surjective ring homomorphism $\deg : K_0(G) \to \mathbb{Z}$.

5. Show that $[V]$ is a unit in $K_0(G)$ if and only if V is 1-dimensional, and that $[V^*]$ is the inverse of $[V]$ if $\dim_{\mathbb{C}} V = 1$.

Remark: The ring $K_0(G)$ is called the *Grothendieck group of* $\mathrm{Rep}_K(G)$ (when considered as an additive group) or the 0-*th term of the K-theory of* $\mathrm{Rep}_K(G)$. More generally, one can define $K_0(\mathcal{C})$ for many other categories $\mathcal{C}$, including all abelian categories.