Exercises for Algebra 2
List 13 To hand in at 16.11.2020

Exercise 1.
Let G be a finite group of order n with exponent

$$\epsilon \;=\; \exp(G) \;=\; \operatorname{lcm}\{\operatorname{ord}(g) \mid g \in G\}.$$

1. Show that ϵ is the smallest positive integer such that $g^\epsilon = e$ for all $g \in G$.
2. Show that ϵ is a divisor of n. Find an example of a group G for which $\epsilon \neq n$.
3. Let K be a field that contains a primitive root of unity of order ϵ. Show that n is invertible in K. Conclude that K contains a primitive n-th root of unity if K is in addition algebraically closed.

Exercise 2.
Let G be a finite group and $m = \#G^{\mathrm{ab}}$. Show that G has precisely m one dimensional simple characters. Conclude that if G is abelian, then all simple characters are one dimensional.

Exercise 3.
Determine all simple characters of A_4, which can be done as follows.

1. Show that A_4 has 4 conjugacy classes and determine representatives for each class.
2. Show that A_4^{ab} is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and conclude that A_4 has precisely 3 one-dimensional characters χ_1, χ_2 and χ_3. Determine these characters.
3. Show that A_4 has precisely one more simple character χ_4, which is of dimension 3. Determine χ_4. Can you find a representation with character χ_4?

Exercise 4.
Determine all simple characters of the dihedral group D_5 with 10 elements, which can be done as follows.

1. Show that D_5 has 4 conjugacy classes and determine representatives for each class.
2. Show that D_5^{ab} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and conclude that D_5 has precisely 2 one-dimensional characters χ_1 and χ_2. Determine these characters.
3. Show that the action of D_5 on the regular pentagon defines a 2-dimensional representation and calculate its character χ_3.
4. Compute the last character as the difference to the regular character.

*__Exercise 5__ (continuation of Exercise 5 from List 12).

Let G be a finite group and $K_0(G)$ the Grothendieck group of G.

1. Show that the association $[V] \mapsto \chi_V$ is defines a (well-defined) injective group homomorphism $\psi : K_0(G) \to \mathbb{C}^{C(G)}$ (w.r.t. the addition in $K_0(G)$). Conclude that $(K_0(G), +)$ is a free abelian group of rank $\#C(G)$.

2. Endow $\mathbb{C}^{C(G)}$ with the product that takes multiplies class functions valuewise, i.e. by the formula $\alpha \cdot \beta(g) = \alpha(g) \cdot \beta(g)$. Show that ψ is a ring homomorphism with respect to this product, and that it extends to an isomorphism $K_0(G) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}^{C(G)}$ of $\mathbb{C}$-algebras.

3. Let $V_1, \ldots, V_s$ be the irreducible representations of G, up to isomorphism, where V_1 is the 1-dimensional trivial representation. Consider for $i, j \in \{1, \ldots, s\}$ the decomposition $V_i \otimes V_j = \bigoplus_{k=1}^{s} V_k^{\oplus m_{i,j,k}}$ into irreducible factors and define the polynomial
$$P_{i,j} \;=\; T_i T_j - \sum_{k=1}^{s} m_{i,j,k} T_k$$
in $\mathbb{Z}[T_1, \ldots, T_s]$. Let I be the ideal of $\mathbb{Z}[T_1, \ldots, T_s]$ that is generated by the polynomials $P_{i,j}$ (for $i, j = 1, \ldots, r$) and $T_1 - 1$. Show that the association $T_i \mapsto [V_i]$ defines an isomorphism $\mathbb{Z}[T_1, \ldots, T_s]/I \to K_0(G)$ of rings.