

Exercises for Algebra 2
List 3

To hand in at 7.9.2020

Exercise 1.

Let $f = T^6 + T^3 + 1 \in \mathbb{Q}[T]$ and $L = \mathbb{Q}[T]/(f)$. Show that f is irreducible and find all field homomorphisms $L \rightarrow \mathbb{C}$. Is L/\mathbb{Q} normal?

Hint: f divides $T^9 - 1$.

Exercise 2.

Proof Fermat's little theorem: If K is a field of characteristic p , then $(a+b)^p = a^p + b^p$. Conclude that $\text{Frob}_{p^n} : K \rightarrow K$ with $\text{Frob}_{p^n}(a) = a^{p^n}$ is a field endomorphism of K .

Remark: Frob_p is called the *Frobenius homomorphism in characteristic p* .

Exercise 3.

Let $\mathbb{F}_p(x)$ be the quotient field of the polynomial ring $\mathbb{F}_p[x]$ in the indeterminant x , i.e. $\mathbb{F}_p(x) = \{f/g \mid f, g \in \mathbb{F}_p[x] \text{ and } g \neq 0\}$.

1. Show that $f = T^p - x$ is irreducible over $\mathbb{F}_p(x)$.

Hint: For a direct calculation, use the factorization of f over $\mathbb{F}_p(\sqrt[p]{x})$; or you can apply the Eisenstein criterium to show that f is irreducible in $\mathbb{F}_p[x, T]$ and conclude with the help of Gauss' lemma that f is irreducible in $\mathbb{F}_p(x)[T]$.

2. Show that f is not separable over $\mathbb{F}_p(x)$.

Hint: Use Fermat's little theorem.

3. Conclude that $\mathbb{F}_p(\sqrt[p]{x})/\mathbb{F}_p(x)$ is not separable. Is $\mathbb{F}_p(\sqrt[p]{x})/\mathbb{F}_p(x)$ normal?

Exercise 4.

Let L/K be a finite field extension and E the separable closure of K in L . Show that $[L : K]_s = [E : K]$ and $[L : E]_s = 1$. Conclude that the separable degree $[L : K]_s$ is a divisor of $[L : K]$.

***Exercise 5.** ¹

1. Find a finite separable (but not normal) field extension L/K that does not satisfy the Galois correspondence.
2. Find a finite normal (but not separable) field extension L/K that does not satisfy the Galois correspondence.
3. Find a normal and separable (but not finite) field extension L/K that does not satisfy the Galois correspondence.

***Exercise 6.** (not to hand in)

Recall the proofs of the Eisenstein criterium and Gauss' lemma.

¹The starred exercises are bonus exercises. It is encouraged to work on these exercises and to hand in solution. The resulting points count as a bonus.