Exercises for Algebra 2
List 5 To hand in at 21.9.2020

Exercise 1.
Let

$$0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 0$$

be a short exact sequence of groups. Show that N and Q are solvable if and only if G is solvable.

Exercise 2. Find all composition series and their factors for the dihedral group

$$D_6 = \langle r, s \,|\, r^6 = s^2 = (rs)^2 = e \rangle.$$

Exercise 3.
Let ζ_{12} be a primitive 12-th root of unity. What is $\mathrm{Gal}(\mathbb{Q}(\zeta_{12}/\mathbb{Q}))$? Find primitive elements for all subfields E of $\mathbb{Q}(\zeta_{12})$.

Exercise 4.
Let p be a prime number and $n \geq 1$ and $\zeta \in \mathbb{F}_{p^n}$ a generator of $\mathbb{F}_{p^n}^\times$. Exhibit an embedding $i : \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \to (\mathbb{Z}/(p^n-1)\mathbb{Z})^\times$ and conclude that n divides $\varphi(p^n-1)$. Can you find a proof for $n|\varphi(p^n-1)$ that does not use Galois theory?

***Exercise 5.**
Show that there is an n_i for $i = 1, 2, 3$ such that the following fields E_i are contained in $\mathbb{Q}(\zeta_{n_i})$. What are the smallest values for n_i?

1. $E_1 = \mathbb{Q}(\sqrt{2})$;
2. $E_2 = \mathbb{Q}(\sqrt{3})$;
3. $E_3 = \mathbb{Q}(\sqrt{-3})$.

Find all $n \geq 1$ such that $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] \leq 3$. Conclude that $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ is not generated by roots of unities over $\mathbb{Q}$.

Hint: Try to realize $\sqrt{2}$ and $\sqrt{3}$ as the side length of certain rectangular triangles. Which angles occur?