Exercises for Algebra 2
List 7 To hand in at 5.10.2020

Exercise 1.
Let K be a field and L the splitting field of a cubic polynomial f over K. Assume that $\zeta_3 \in L$ and that L/K is separable. Show that there is a subfield E of L such that $K \subset E \subset L$ is a tower of elementary radical extensions (with possibly $L = E$ or $E = K$). In which situations are E/K and L/E cyclotomic, Kummer and Artin-Schreier? What are E and L if $K = \mathbb{Q}$ and $f = T^3 - b \in \mathbb{Q}[T]$?

Exercise 2.
Let K be a field and L the splitting field of a polynomial f over K of degree 4 or less. Show that L/K is solvable if it is separable.

Exercise 3.
Let $L/\mathbb{Q}$ be a cubic solvable extension, i.e. $[L : \mathbb{Q}] = 3$. Show that $L/\mathbb{Q}$ is not radical. Show that such an extension exists.

Hint: Show that if $L/\mathbb{Q}$ was radical, it must contain ζ_3. Lead this to a contradiction.

Exercise 4.
Let p be a prime number and f an irreducible polynomial of degree p over $\mathbb{Q}$ that has exactly 2 complex roots. Let L be the splitting field of f over $\mathbb{Q}$. Show that $\mathrm{Gal}(L/\mathbb{Q}) \simeq S_p$.

***Exercise 5.**
Find a finite Galois extension $L/\mathbb{Q}$ with Galois group A_5, a primitive element $a \in L$ and its minimal polynomial f over $\mathbb{Q}$.