Exercises for Algebra 2
List 8 To hand in at 12.10.2020

Exercise 1.
Let K be a subfield of $\mathbb{C}$ and a a root of $T^2 - b \in K[T]$. Show that every element of $K(a)$ is constructible over K. Use this to explain the relationship between the two definitions of constructible numbers from sections 1.1 and 4.6 of the lecture.

Exercise 2.
Which of the following elements are constructible over $\mathbb{Q}$?

1. $\sqrt{2} + \sqrt{3}$, $\sqrt{2} \cdot \sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[4]{3}$.
2. ζ_n for $n = 1, \dots, 20$.
3. $\zeta_7 + \zeta_7^{-1}$, $\zeta_7 + \zeta_7^2 + \zeta_7^4$.

Let a be any of the above elements and L the normal closure of $\mathbb{Q}(a)/\mathbb{Q}$. Calculate $N_{L/\mathbb{Q}}(a)$ and $\mathrm{Tr}_{L/\mathbb{Q}}(a)$.

Exercise 3.

1. Find a radical tower for the splitting field L of $f = T^8 - 7$ over $\mathbb{Q}$. Prove all your assertions!
2. Show that every root of f is constructible over $\mathbb{Q}$. Show that the roots of $T^7 - 7$ are not constructible.

Exercise 4.
Find a normal basis for $\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}$ and all its intermediate extensions $E/\mathbb{Q}$.

***Exercise 5.**

1. Describe a non-trivial element σ of $G = \mathrm{Gal}(\mathbb{F}_{27}/\mathbb{F}_3)$. How many orbits does the action of G on $\mathbb{F}_{27}$ have? What is the cardinality of each orbit?
2. Find all cubic monic irreducible polynomials $f = T^3 + c_2T^2 + c_1T + c_0$ in $\mathbb{F}_3[T]$.
3. For which of these cubic polynomials does the set of roots $\{a_1, a_2, a_3\}$ form a normal basis for $\mathbb{F}_{27}/\mathbb{F}_3$?
4. Prove that all normal bases for $\mathbb{F}_{27}/\mathbb{F}_3$ are of this form.