Seminar on $\mathbb{F}_1$-Geometry: Tropicalization as a Base Change I

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Outline

Motivation

Consider a field k together with a valuation $\nu : k^{\times} \to \mathbb{R}_{\geq 0}$. Given a variety X/k there are several topological spaces we can attach to X.

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- If we fix an embedding in a toric variety: A Kajiwara-Payne tropicalization $\mathrm{Trop}^{KP}(X)$.
- A Thuillier analytification $X^{\beth}$.
- If we fix an embedding in a toroidal variety or a log scheme: An Ulrisch tropicalization $\mathrm{Trop}^{U}(X)$.

Motivation

For each of these spaces we can find an Ordered Blue scheme

$$\mathbf{Y} \longrightarrow \mathrm{Spec}(\mathbf{k})$$

depending in the variety X and its extra data such that we can recover the space as *rational points in the tropicalization* $\mathbf{Y} \otimes_{\mathbf{k}} \mathbf{T}$ of $\mathbf{Y}$.

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- $\mathbf{Y}$ endows each tropicalization with extra structure.
- We can tropicalize with respect to more general valuations using the same toolbox.
  For example, with $\mathbf{Y} \otimes_{\mathbf{k}} \mathbf{T}^k$ one recovers Foster Ranganathan analytification/tropicalization.

Ordered Blueprints

Recall that an *ordered blueprint* is a tuple $B = (B^\bullet, B^+, \leq)$ where:

- B^+ is a commutative semiring.
- $B^\bullet$ is a multiplicative submonoid of B^+ containing $0, 1$ and which generates B^+ as a semiring.
- $\leq$ is an additive and multiplicative partial order on B^+.

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We consider $B^\bullet$ as the underlying set of the ordered blueprint. In particular, to see B as a topological space we put a topology on $B^\bullet$.

Ordered Blueprints

Moreover, we have defined two functors

$$\begin{aligned} \mathcal{G} : \mathrm{CMon} &\longrightarrow \mathrm{OBlpr} \\ A &\longmapsto \mathcal{G}(A) = (A, \mathbb{N}[A \setminus \{0\}], =) \end{aligned}$$

$$\begin{aligned} \mathcal{F} : \mathrm{SRing} &\longrightarrow \mathrm{OBlpr} \\ R &\longmapsto \mathcal{F}(R) = (R, R, =) \end{aligned}$$

and for a fixed ordered blueprint B a functor

$$\begin{aligned} \mathrm{CMon} &\longrightarrow \mathrm{OBlpr}/B \\ A &\longmapsto B[A] = (B[A]^{\bullet}, B[A]^{+}, \leq) \end{aligned}$$

Multiplicative Monomial Ordered Blueprints

We now introduce a new functor defined as

$$\begin{aligned} (\quad)^{\text{mon}} : \text{Sring} &\longrightarrow \text{OBlpr} \\ R &\longrightarrow R^{\text{mon}} = \mathbf{R} = (\mathbf{R}^{\bullet}, (\mathbf{R}^{\bullet})^{+}, \leq) \end{aligned}$$

where

- $\mathbf{R}^{\bullet}$ is the multiplicative monoid of R,
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This is the *associated monomial ordered blueprint* of R. $(\quad)^{\mathrm{mon}}$ extends easily to morphisms by functoriality of the entries.

Tensor Product of Ordered Blueprints

Given two morphisms $B \to C$ and $B \to D$ of ordered blueprints we can construct the following ordered blueprint.

Definition: tensor products

Given two morphisms $B \to C$ and $B \to D$ of ordered blueprints, their *tensor product* is the ordered blueprint

$$C \otimes_B D = ((C \otimes_B D)^\bullet, (C \otimes_B D)^+, \leq), \quad \text{where}$$

- $(C \otimes_B D)^+ = C^+ \otimes_{B^+} D^+$ is the tensor product of the respective semirings.
- $(C \otimes_B D)^\bullet$ is the set of pure tensors $c \otimes d$ for $c \in C, d \in D$
- $\leq$ is the partial order generated by

$$\{x \otimes 1 \leq y \otimes 1 \mid x \leq y \in C^+\} \cup \{1 \otimes x \leq 1 \otimes y \mid x \leq y \in D^+\}$$

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Let now B be a blueprint and X/B an ordered blue scheme over B, then if $\mathcal{U} = (U_i)_{i \in I}$ is an affine presentation with $U_i = \operatorname{Spec} C_i$ and if $B \rightarrow D$ is a morphism of blueprints, the basechange $X \otimes_B D$ is the blue scheme

$$\operatorname{colim}_{i \in I} \operatorname{Spec} C_i \otimes_B D.$$

Tensor Product of Ordered Blueprints

As the tensor product is a pushout, if C is a blueprint over B, T is an extension of B and D is a blueprint over T then

$$\mathrm{Hom}_B(C, D) = \mathrm{Hom}_T(C \otimes_B T, D)$$

More generally, if X is a scheme over B

$$(X \otimes_B T)(D) = X(D)$$

The Tropical Hyperfield

The *tropical hyperfield* $\mathbb{T}^{\mathrm{hyp}}$ is the hyperring with underlying set is $\mathbb{R}_{\geq 0}$ together with the usual product and the *hyperaddition*

$$a \boxplus b = \begin{cases} \{\max\{a, b\}\} & \text{if } a \neq b \\ [0, a] & \text{if } a = b \end{cases}$$

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However, there is a natural way to embed the category of hyperrings into OBlpr.

We use this to work with the tropical hyperfield as a blueprint without worring about its original definition

The Tropical Hyperfield

As a blueprint, the tropical hyperfield is given by

$$\mathbf{T} = (\mathbb{R}_{\geq 0}, \mathbb{N}[\mathbb{R}_{\geq 0}], \leq)$$

where $\leq$ is the order generated by $\{c \leq a + b \text{ in } \mathbf{T}^\bullet \mid c \in a \boxplus b\}$.

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That is $c \leq a + b$ if either $a \neq b$ and $c = \max\{a, b\}$ or $c \leq a = b$. In other words, if $\max\{a, b, c\}$ is attained at least twice.

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Remark

More generally, given $a, b_1, \ldots, b_n \in \mathbf{T}$ we have $a \leq \sum_{j=1}^n b_j$ in T iff $\max\{a, b_1, \ldots, b_n\}$ is attained at least twice.

Valuations

Let R be a ring, a nonarchimedean seminorm is a map

$$\nu : R \to \mathbb{R}_{\geq 0}$$

such that

1. $\nu(0) = 0$ and $\nu(1) = 1$
2. $\nu(ab) = \nu(a)\nu(b)$
3. $\nu(a + b) \leq \max\{\nu(a), \nu(b)\}$

Lemma

For a semi norm $\nu : R \to \mathbb{R}_{\geq 0}$ and an equality $a = \sum_{j=1}^{n} b_j$ in R, the maximum of $\{\nu(a), \nu(b_1), \dots, \nu(b_n)\}$ is attained at least twice.

Lemma

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Proof. Notice that $\nu(a) = \nu(-1)\nu(a) = \nu(a)$. If we now denote $b_{n+1} = -a$ we have $\sum_{j=1}^{n+1} b_j = 0$. If the maximum is attained once, say in b_1, we would have that

$$\begin{aligned}\nu(b_1) &= \nu(-b_2 - \cdots - b_{n+1}) \\ &\leq \max\{\nu(b_2), \ldots, \nu(b_{n+1})\} \\ &< \nu(b_1)\end{aligned}$$

which is not possible.

Valuations as Morphisms

Now we are in conditions to state the main result of the talk

Theorem

For a ring R the following map is a bijection

$$\begin{aligned} \Phi: \ \mathrm{Hom}(\mathbf{R}, \mathbf{T}) &\longrightarrow \{\text{Nonarchimedean seminorms on } R\} \\ \boldsymbol{\nu}: \mathbf{R} \to \mathbf{T} &\longmapsto \boldsymbol{\nu}^{\bullet}: R \to \mathbb{R}_{\geq 0} \end{aligned}$$

Proof. Notice that $\mathbf{R}^\bullet = R$ and $\mathbf{T}^\bullet = \mathbb{R}_{\geq 0}$, so the domain and codomain are correct. Now, let us see that if $\boldsymbol{\nu}$ is a morphism then $\boldsymbol{\nu}^\bullet$ is a seminorm.

Proof. Notice that $\mathbf{R}^\bullet = R$ and $\mathbf{T}^\bullet = \mathbb{R}_{\geq 0}$, so the domain and codomain are correct. Now, let us see that if $\boldsymbol{\nu}$ is a morphism then $\boldsymbol{\nu}^\bullet$ is a seminorm.

As $\boldsymbol{\nu}^\bullet$ is a morphism of monoids with zero we have $\boldsymbol{\nu}^\bullet(0) = 0$, $\boldsymbol{\nu}^\bullet(1) = 1$ and $\boldsymbol{\nu}^\bullet(ab) = \boldsymbol{\nu}^\bullet(a)\boldsymbol{\nu}^\bullet(b)$.

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Moreover, if $c = a + b$ in R then $c \leq a + b$ in $\mathbf{R}^\bullet$ and thus $\boldsymbol{\nu}^\bullet(c) \leq \boldsymbol{\nu}^\bullet(a) + \boldsymbol{\nu}^\bullet(c)$ in $\mathbf{T}$.

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Hence $\boldsymbol{\nu}^\bullet$ the maximum between $\boldsymbol{\nu}^\bullet(a), \boldsymbol{\nu}^\bullet(b)$ and $\boldsymbol{\nu}^\bullet(c)$ is attained at least twice so $\boldsymbol{\nu}(c) \leq \max\{\boldsymbol{\nu}(a), \boldsymbol{\nu}(b)\}$.

Conversely, take a seminorm $\nu : R \to \mathbb{R}_{\geq 0}$. Then ν is a morphism of monoids with zero so it defines a map $\boldsymbol{\nu}^{\bullet} : \mathbf{R}^{\bullet} \to \mathbf{T}^{\bullet}$.

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Now consider the unique extension $\boldsymbol{\nu}^+ : \mathbf{R}^+ \to \mathbf{T}^+$ which is additive. We just have to prove that $\boldsymbol{\nu}^+$ is order preserving. For this we can show it over the generators $c \leq a + b$.

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Now consider the unique extension $\boldsymbol{\nu}^{+} : \mathbf{R}^{+} \to \mathbf{T}^{+}$ which is additive. We just have to prove that $\boldsymbol{\nu}^{+}$ is order preserving. For this we can show it over the generators $c \leq a + b$.

From $c \leq a + b$ in R we get $c = a + b$. Then by the Lemma above we have that the maximum between $\nu(c), \nu(a), \nu(b)$ is attained at least twice. As we saw before this is equivalent to $\nu(c) \leq \nu(a) + \nu(b)$. $\square$

Tropicalization of Ordered Blue Schemes

In general, we introduce the following concept.

Definition

Let $\nu : k \to \mathbb{R}_{\geq 0}$ be a valued field and Y an ordered blue scheme over $\mathbf{k}$. Then, the tropicalization of Y along ν is the blue scheme over $\mathbf{T}$ given by

$$\mathrm{Trop}_{\nu}(Y) = Y \otimes_{\mathbf{k}} \mathbf{T}$$

Here we are using the associated morphism $\boldsymbol{\nu} : \mathbf{k} \to \mathbf{T}$ from the Theorem to see $\mathbf{T}$ over $\mathbf{k}$.

Tropicalization of Ordered Blue Schemes

Notice that by the universal property of the base change, for any ordered blue scheme Y over $\mathbf{k}$

$$Trop_{\nu}(Y)_{\mathbf{k}}(\mathbf{T}) = Y(\mathbf{T})$$

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In Particular, if we take $Y = \mathbf{X}$ for X an affine algebraic variety X over k, we have by the theorem

$$Trop_{\nu}(Y)_{\mathbf{k}}(\mathbf{T}) \cong X^{\mathrm{an}}$$