Seminar on $\mathbb{F}_1$-Geometry: Tropicalization as a Base Change II

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Outline

Review last week

Last time we saw

- The functor of *associated monomial oblpr*

$$\begin{aligned} \mathrm{SRings} &\longrightarrow \mathrm{OBlpr} \\ R &\longmapsto \mathbf{R} = (R, \mathbb{N}[R \setminus 0], \leq) \end{aligned}$$

  where R is the multiplicative monoid underlying R and $c \leq a + b$ iff $c = a + b$.

- The tropical hyperfield blueprint $\mathbf{T}$ given by

$$\mathbf{T} = (\mathbb{R}_{\geq 0}, \mathbb{N}[\mathbb{R}_{>0}], \leq)$$

  where $c \leq a + b$ iff $c \in a \boxplus b$.

Review last week

- The bijection

$$\begin{array}{rcl} \Phi: \ \mathrm{Hom}(\mathbf{R},\mathbf{T}) & \longrightarrow & \{\text{Nonarchimedean seminorms } R \to \mathbb{R}_{\geq 0}\} \\ \boldsymbol{\nu}: \mathbf{R} \to \mathbf{T} & \longmapsto & \boldsymbol{\nu}^{\bullet}: R \to \mathbb{R}_{\geq 0} \end{array}$$

- Given a blue scheme Y over k and a valuation $\nu : k \to \mathbb{R}_{\geq 0}$, its tropicalization is

$$\mathrm{Trop}_{\nu}(Y) = Y \otimes_{\mathbf{k}} \mathbf{T}$$

and if $Y = \mathrm{Spec}(B)$ then

$$\mathrm{Trop}_{\nu}(Y)(\mathbf{T}) = \mathrm{Hom}_{\mathbf{k}}(B, \mathbf{T})$$

Objective

Today we will see

- How to put a topology on a set of rational points, such as $\mathrm{Trop}_\nu(Y)(\mathbf{T})$.
- How to see the Kajiwara-Payne tropicalization and Berkovich analytification as rational points.
- How to obtain a blue version of Giansiracusa Bend relations.
- A comparison between the Giansiracusa approach and the hyperfield approach.

Topology on Rational Points

Let $X = \operatorname{Spec} B$ be an affine ordered blue $\mathbf{T}$-scheme.

Affine topology

The *affine topology* on $X(\mathbf{T}) = \operatorname{Hom}_{\mathbf{T}}(B, \mathbf{T})$ is the coarsest topology for which the evaluations maps

$$\begin{aligned} \mathrm{ev}_a : X(\mathbf{T}) &\longrightarrow \mathbf{T} \\ f : B \to T &\longmapsto f(a) \end{aligned}$$

are continuous where we endow $\mathbf{T} = \mathbb{R}_{\geq 0}$ with its euclidean topology (equiv. order topology).

Topology on Rational Points

The affine topology is functorial and behaves well under identifications, more precisely

Theorem

Let X, Y be $\mathbf{T}$- blue schemes, then

1. The canonical map $\mathbb{A}^1_{\mathbf{T}}(\mathbf{T}) \to \mathbf{T}$ is a homeomorphism.
2. The canonical map $(X \times Y)(\mathbf{T}) \to X(\mathbf{T}) \times Y(\mathbf{T})$ is a homemorphism.
3. For a $\mathbf{T}$-morphism $\varphi : Y \to X$, the induced map $\varphi_{\mathbf{T}} : Y(\mathbf{T}) \to X(\mathbf{T})$ is continuous.
   It is open/closed when φ is an open/closed immersion.

Topology on Rational Points

If $X \hookrightarrow \mathbb{A}^n_k$ is a variety then $\mathrm{Trop}_v(\mathbf{X}) \hookrightarrow \mathbb{A}^n_{\mathbf{T}}$ is a closed immersion of blue schemes so $\mathrm{Trop}_v(\mathbf{X})(\mathbf{T}) \subseteq \mathbb{A}^n_{\mathbf{T}}(\mathbf{T}) = \mathbf{T}^n$ is a closed subset.

In the non affine case, given a scheme X we can take an affine presentation $X = \bigcup_i U_i$, then

$$X(\mathbf{T}) = \bigcup U_i(\mathbf{T})$$

and we put the topology on $X(\mathbf{T})$ given by the gluing (colimit) of the topologies on $U_i(\mathbf{T})$. This is called the *fine topology*.

Berkich Spaces

Recall that if $v : k \to \mathbb{R}_{\geq 0}$ is a non-archimedean absolute value, R is a k-algebra and $X = \operatorname{Spec}(R)$ then

$$X^{\mathrm{an}} = \{\text{non-archimedean seminorms } w : R \to \mathbb{R}_{\geq 0} \text{ with } w|_k = v\}$$

with the coarsest topology such that

$$\begin{aligned} \mathrm{ev}_a : X^{\mathrm{an}} &\longrightarrow \mathbb{R}_{\geq 0} \\ w &\longmapsto w(a) \end{aligned}$$

is continuous.

Berkich Space as Rational Points

Now, the bijection of last week

$$\begin{aligned} \Phi : \ \mathrm{Hom}(\mathbf{R}, \mathbf{T}) &\longrightarrow \{\text{NA seminorms } R \to \mathbb{R}_{\geq 0}\} \\ \boldsymbol{\nu} : \mathbf{R} \to \mathbf{T} &\longmapsto \boldsymbol{\nu}^{\bullet} : R \to \mathbb{R}_{\geq 0} \end{aligned}$$

restricts to a bijection

$$\mathrm{Hom}_k(\mathbf{R}, \mathbf{T}) \longrightarrow \{\text{NA seminorms } w : R \to \mathbb{R}_{\geq 0} \text{ with } w|_k = v\}$$

or, changing notations,

$$\Phi^{\mathrm{an}} : \mathrm{Trop}_v(\mathbf{X})(\mathbf{T}) \to X^{\mathrm{an}}$$

which is a homeomorphism with the definition of the topologies in both sides.

Kajiwara-Payne Tropicalization

Let $T = \text{Spec}(k[A])$ be an affine toric variety, $X = \text{Spec}(R)$ a variety and $\iota : X \to T$ a closed embedding. Then we have a surjection $\iota^\sharp : k[A] \to R$ with kernel I.

Consider

$$X^{\text{trop}} = \{f \in \mathbb{R}^A_{\geq 0} \,|\forall p = \sum c_a a \in I,$$
$$\text{the maximum in } \{v(c_a)a\} \text{ is attained twice } \}$$

If $A = \sigma^\vee \cap M$ then $\mathbb{R}^A_{\geq 0} = \overline{\sigma}$ and $X^{\text{trop}} \subseteq \overline{\sigma}$ is a corner locus.

The Kajiwara-Payne tropicalization is the map

$$X^{\text{an}} \longrightarrow X^{\text{trop}}$$
$$w \longmapsto w \circ \iota^\sharp$$

Kajiwara-Payne Tropicalization as Rational points

The morphism $\iota^\sharp : k[A] \to R$ induces a morphism. Now define a blueprint B by

$$B = k[A] /\!\!/ \langle c_a a \leq \sum c_j b_j \mid \pi(c_a a) \leq \sum \pi(c_j b_j) \in \mathbf{R} \rangle$$

that is, $B^\bullet = \{\pi(ca) \mid c \in k, a \in A\}$, $B^+ = \mathbb{N}[B^\bullet \setminus \{0\}]$ and whose order is generated by the relations

$$c_a a \leq \sum c_j b_j \text{ when } \iota^\sharp(c_a a) = \sum \iota^\sharp(c_j b_j) \in R.$$

Then there is a bijection

$$\Phi^{\mathrm{trop}} : X^{\mathrm{trop}} \longrightarrow \mathrm{Trop}_v(Y)(\mathbf{T})$$

Kajiwara-Payne Tropicalization as Rational points

Example: Consider

$$X = \operatorname{Spec} k[T_1, T_2]/(T_1 + T_2 + 1) \hookrightarrow T = \operatorname{Spec} k[T_1, T_2]$$

then $B = \mathbf{k}[T_1, T_2] /\!\!/ \langle \text{generators} \rangle$ where the generators are

$0 \leq T_1 + T_2 + 1, \; -T_1 \leq T_2 + 1, \; -T_2 \leq T_1 + 1, \; -1 \leq T_1 + T_2$

Then $B \otimes_k \mathbf{T} = \mathbf{T}[T_1, T_2] /\!\!/ \langle \text{generators} \rangle$ for the same generators. Then the map

$$\Psi^{\mathrm{trop}} : Hom(B \otimes_k \mathbf{T}, \mathbf{T}) \longrightarrow \mathbf{T}^2$$
$$f \longmapsto (f(T_1 \otimes 1), f(T_2 \otimes 2))$$

has as image exactly

$$X^{\mathrm{trop}} = \{(a_1, a_2) \in \mathbf{T}^2 \mid \max\{a_1, a_2, 1\} \text{ is attained twice}\}$$

so it is the inverse of the map above.

Kajiwara-Payne Tropicalization as Rational points

In this way we get a commutative diagram.

$$\begin{array}{ccc} X^{\mathrm{an}} & \xrightarrow{\mathrm{trop}} & X^{\mathrm{trop}} \\ \Big\downarrow{\scriptstyle \Phi^{\mathrm{an}}} & & \Big\downarrow{\scriptstyle \Phi^{\mathrm{trop}}} \\ \mathrm{Trop}_{\mathbf{v}}(\mathbf{X})(\mathbf{T}) & \xrightarrow{\beta_*} & \mathrm{Trop}_{\mathbf{v}}(Y)(\mathbf{T}) \end{array}$$

where the morphism in the lower part is induced by the inclusion $\beta : B \to \mathbf{R}$

Bend Relations

Let A be an arbitrary monoid, $v : k \to \mathbb{R}$ a valued field, $T = \operatorname{Spec} k[A]$ and $X = \operatorname{Spec} R \hookrightarrow T$ a closed immersion. Then we have a surjection $\iota^\sharp : K[a] \to R$ with kernel I.

The *bend relations* on I is the congruence in $\mathbb{R}_{\geq 0}[A]$ given by

$$\mathrm{bend}_{v,\iota}(I) = \{f^{\mathrm{trop}} \sim f_{a_0}^{\mathrm{trop}} \mid f \in I, a_0 \in \mathsf{Supp}\{f\}\}$$

where for $f = \sum c_a a \in k[A]$ we define $f^{\mathrm{trop}} = \sum v(c_a)a$ and $f_{a_0}^{\mathrm{trop}} = \sum_{A\setminus\{a_0\}} v(c_a)a$.

Moreover, the Giansiracusa bend of R is the semiring

$$\mathrm{Bend}_{v,\iota}^{GG}(R) = \mathbb{R}_{\geq 0}[A]/\mathrm{bend}_{v,\iota}(I)$$

Bend Relations as a Blue Scheme

We can considering the Giansiracusa bend by considering the quotient above as a blueprint quotient

$$\mathrm{Bend}_{v,\iota}(R) = \mathbb{T}[A] /\!\!/ \mathrm{bend}_{v,\iota}(I)$$

In that way $\mathrm{Bend}_{v,\iota}(R)^{+} = \mathrm{Bend}_{v,\iota}^{GG}(R)$, the order is trivial and

$$\mathrm{Bend}_{v,\iota}(R)^{\bullet} = \{ta \in \mathbb{R}_{\geq 0}[A] \mid t \in \mathbb{R}_{\geq 0}, a \in A\}$$

The Giansiracusa bend of X is $\mathrm{Spec}(\mathrm{Bend}_{v,\iota}(R))$

Bend relations as a blue Scheme

In the same way as for semirings. If $\iota : X \hookrightarrow \operatorname{Spec} k[A]$ is a closed immersion into a toric variety then

$$\mathrm{Bend}_{v,\iota}(X)(\mathbb{T}) \cong X^{\mathrm{trop}}$$

and if $\widehat{\iota} : X \hookrightarrow \operatorname{Spec} k[R \setminus 0]$ then

$$\mathrm{Bend}_{v,\widehat{\iota}}(X)(\mathbb{T}) \cong X^{\mathrm{an}}.$$

Moreover we have a commutative diagram

$$\begin{array}{ccc}
X^{\mathrm{an}} & \xrightarrow{\quad\mathrm{trop}\quad} & X^{\mathrm{trop}} \\
\simeq\big\downarrow & & \big\downarrow\simeq \\
\mathrm{Bend}_{v,\widehat{\iota}}(X)(\mathbb{T}) & \xrightarrow{\quad\beta_*\quad} & \mathrm{Bend}_{v,\iota}(X)(\mathbb{T}).
\end{array}$$

From the tropical hyperfield to the tropical semifield

Recall that $\mathbb{T}$ is the *tropical semiring*, that is, $\mathbb{R}_{\geq 0}$ with the usual multiplication and maximum as the addition, as an ordered blueprint it is given by

$$\mathbb{T} = (\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}, =)$$

Meanwhile $\mathbf{T}$ is the *tropical hyperring*, which as a blueprint is

$$\mathbf{T} = (\mathbb{R}_{\geq 0}, \mathbb{N}[\mathbb{R}_{>0}], \geq)$$

Now we will see a nice way to go from $\mathbf{T}$ to $\mathbb{T}$. For this we have to introduce the following functors

From the tropical hyperfield to the tropical semifield

1. Associated Idempotent semiring: Takes a blueprint and sends it to the universal blueprint such that $1 + 1 = 1$.

$$\begin{aligned} \mathrm{OBlpr} &\longrightarrow \mathrm{OBlpr} \\ B &\longmapsto B^{\mathrm{idem}} = B /\!\!/ \langle 1 + 1 \equiv 1 \rangle \end{aligned}$$

2. Associated totally positive blueprint: Takes a blueprint and sends it to the universall blueprint such that $0 \leq 1$.

$$\begin{aligned} \mathrm{OBlpr} &\longrightarrow \mathrm{OBlpr} \\ B &\longmapsto B^{\mathrm{pos}} = B /\!\!/ \langle 0 \leq 1 \rangle \end{aligned}$$

3. Associated algebraic blueprint/Algebraic core: Takes an ordered blueprint and forgets its order structure.

$$\begin{aligned} \mathrm{OBlpr} &\longrightarrow \mathrm{OBlpr} \\ B &\longmapsto B^{\mathrm{core}} = (B^{\bullet}, B^{+}, =) \end{aligned}$$

From the tropical hyperfield to the tropical semifield

Proposition

The identity map $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ lifts to a morphism

$$\mathbf{T} \longrightarrow \mathbb{T}^{\mathrm{pos}}$$

which induces an isomorphism

$$\mathbf{T}^{\mathrm{idem}} \longrightarrow \mathbb{T}^{\mathrm{pos}}$$

in particular, by taking algebraic cores

$$(\mathbf{T}^{\mathrm{idem}})^{\mathrm{core}} \longrightarrow \mathbb{T}$$

Comparison of the semiring and hyperring approach

Let X be a variety and $\iota : X \to k[A]$ an embedding into a toric variety. Then, if we define as before

$$B = k[A] /\!\!/ \langle c_a a \leq \sum c_j b_j \mid \pi(c_a a) \leq \sum \pi(c_j b_j) \in \mathbf{R} \rangle$$

there are isomorphisms

$$\mathrm{Bend}_{v,\iota}(X)^{\mathrm{pos}} \xrightarrow{\sim} \mathrm{Trop}_v(Y)^{\mathrm{idem}}$$
$$\mathrm{Bend}_{v,\iota}(X) \xrightarrow{\sim} (\mathrm{Trop}_v(Y)^{\mathrm{idem}})^{\mathrm{core}}$$

In particular

$$\mathrm{Bend}_{v,\iota}(R) \cong (B \otimes_k \mathbf{T}^{\mathrm{idem}})^+.$$