Algebraic K-theory and Grothendieck–Witt theory of monoid schemes

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Based on joint work [ELY] with Jens Eberhardt (Universität Bonn) and Oliver Lorscheid (IMPA).

- *Group completion in the K-theory and Grothendieck–Witt theory of proto-exact categories.* arXiv:2009.12635.
- *Algebraic K-theory and Grothendieck–Witt theory of monoid schemes.* arXiv:2009.12636.

Main result of the talk

Description of the algebraic K-theory and Grothendieck–Witt theory spaces of an integral monoid scheme X in terms of its Picard group $\mathrm{Pic}(X)$ and pointed monoid $\Gamma(X) = \Gamma(X, \mathcal{O}_X)$ of global functions, thought of as objects with involution in the latter case.

- This recovers and refines earlier results of Deitmar and Chu–Lorscheid–Santhanam [CLS] on K_0 of monoid schemes
- This is the first study of Grothendieck–Witt theory of monoid schemes. In particular, all results in this direction are new.

Background on algebraic K-theory

Algebraic K-theory

- Algebraic analogue of complex topological K-theory
- Systematically developed by Quillen, Waldhausen, Thomason, ...

  Quillen's contribution included a flexible categorical formulation of the theory: the input is an exact category $\mathcal{A}$.

  Key examples:
- R-proj, fg proj. modules over a commutative ring $R \rightsquigarrow K_i(R)$
  - R-proj is *split* exact
- $\mathrm{Vect}(X)$ for scheme $X \rightsquigarrow K_i(X)$
  - $\mathrm{Vect}(X)$ is non-split exact
- $\mathrm{Coh}(X)$ for a nice scheme $X \rightsquigarrow G_i(X)$
  - $\mathrm{Coh}(X)$ is non-split abelian

A definition via Quillen's Q-construction

Let $Q(\mathcal{A})$ be the category with

- Objects: objects of $\mathcal{A}$
- Morphisms $U \to V$: (equivalence classes of) diagrams of the form

$$\begin{tikzcd}
& X \arrow[dl, two heads] \arrow[dr, hook] & \\
U & & V
\end{tikzcd}$$

- Composition: Induced by pullback

$$(V \to W) \circ (U \to V) =$$

$$\begin{tikzcd}
& & X \times_V Y \arrow[dl, two heads] \arrow[dr, hook] & & \\
& X \arrow[dl, two heads] \arrow[dr, hook] & & Y \arrow[dl, two heads] \arrow[dr, hook] & \\
U & & V & & W
\end{tikzcd}$$

Definition (Quillen)

The algebraic K-theory space of $\mathcal{A}$ is the based loop space

$$\mathcal{K}(\mathcal{A}) = \Omega BQ(\mathcal{A}).$$

The higher algebraic K-theory groups of $\mathcal{A}$ are

$$K_i(\mathcal{A}) = \pi_i \mathcal{K}(\mathcal{A}).$$

When $i = 0$,

$$K_0(\mathcal{A}) \simeq \frac{\text{free abelian group on } \mathrm{Iso}(\mathcal{A})}{\langle [V] = [U] + [W] \text{ if } U \hookrightarrow V \twoheadrightarrow W \rangle}.$$

When $i = 1, 2, 3$, the groups $K_i(R)$ agree with those of Bass, Milnor and Gersten.

Grothendieck–Witt theory (a.k.a. Hermitian *K*-theory)

- Algebraic analogue of Atiyah's topological *KR*-theory (which includes *K*-, *KO*- and *KSp*-theory)
- Developed by Knebusch, Quebbemann-Scharlau-Schulte, ..., Karoubi, ..., Schlichting, ...
- The original motivation was to better understand Witt groups of quadratic forms and their global geometric analogues
- Has recently found applications in enumerative geometry (Levine, Hoyois, Kass–Winkelgren, ...), including curve counting (Welschinger invariants), Grothendieck–Lefschetz–Verdier traces

The categorical setting of Grothendieck–Witt theory refines that of K-theory:

- $\mathcal{A}$ an exact category
- $P : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ an exact functor
- $\Theta : \mathrm{id}_{\mathcal{A}} \Rightarrow P^2$ a coherent natural isomorphism

Key (non)-examples:

- $\mathcal{A} = R$-proj with $P = \mathrm{Hom}_R(-, \mathcal{L})$ and $\Theta = \pm\mathrm{ev}$, where $\mathcal{L} \in \mathrm{Pic}(R) \rightsquigarrow GW_i(R; \mathcal{L})$ of [QSS], Karoubi, ...
- $\mathcal{A} = \mathrm{Vect}(X)$ with $P = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{L})$ and $\Theta = \pm\mathrm{ev}$ $\rightsquigarrow GW_i(X; \mathcal{L})$
- $\mathcal{A} = \mathrm{Coh}(X)$ does *not* admit a duality, so there is no Grothendieck–Witt analogue of $G_i(X)$.

Constructions with dualities

- A *symmetric form* in $\mathcal{A}$ is an object $M \in \mathcal{A}$ and an isomorphism $\psi_M : M \to P(M)$ which satisfies $P(\psi_M) \circ \Theta_M = \psi_M$.
- An *isotropic subobject* of (M, ψ_M) is $i : U \hookrightarrow M$ which factors through $U^{\perp} := \ker(M \to P(M) \overset{P(i)}{\twoheadrightarrow} P(U)) \hookrightarrow M$
- The *reduction* of (M, ψ_M) by an isotropic $U \hookrightarrow M$ is $M /\!\!/ U := U^{\perp}/U$ with its induced symmetric form $\psi_{M/\!\!/U}$.

Example: A symmetric form in R-proj is a projective R-module with a non-degenerate R-module homomorphism

$$\psi_M : M \otimes_R M \to R$$

which is symmetric ($\Theta = +\mathrm{ev}$) or skew-symmetric ($\Theta = -\mathrm{ev}$)

Let $Q_h(\mathcal{A})$ be the category with

- Objects: symmetric forms in $\mathcal{A}$
- Morphisms $(M, \psi_M) \to (N, \psi_N)$: diagrams of the form

$$\begin{tikzcd} & V \arrow[dl, two heads, "q"'] \arrow[dr, hook, "i"] & \\ (M, \psi_M) & & (N, \psi_N) \end{tikzcd}$$

with i coisotropic and q presenting M as $N /\!\!/ V^{\perp}$

Definition (Charney–Lee, Uridia)

The Grothendieck–Witt space of $\mathcal{A}$ is

$$\mathcal{GW}(\mathcal{A}) = hofib(BQ_h(\mathcal{A}) \xrightarrow{\text{forget}} BQ(\mathcal{A})).$$

When $i = 0$,

$$GW_0(\mathcal{A}) = \frac{\text{free abelian group on Symm}(\mathcal{A})}{\langle [M] = [H(U)] + [M /\!\!/ U] \text{ if } U \hookrightarrow M \text{ isotropic} \rangle}.$$

K-theory of monoid schemes

What is the correct categorical setting?

Characteristic property of $\mathbb{F}_1$-geometry

Categories appearing in $\mathbb{F}_1$-geometry are typically not additive, but rather have a weaker, partially additive structure (without a universal property). In particular, these categories do not have exact structures.

Example: $\mathcal{A} = \mathbb{F}_1\text{-mod} = \text{set}_*$. Replacing the direct sum is

$$U \oplus V = (U \sqcup V)/\langle 0_U \sim 0_V \rangle.$$

This is not a product:

$$\begin{tikzcd}
 & U \arrow[dl, "id"'] \arrow[dr, "id"] & \\
U & U \oplus U \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & U
\end{tikzcd}$$

There are no diagonal maps!

Non-additive analogues of Quillen's exact categories were defined, and applied to study $\mathbb{F}_1$-geometry, by Deitmar.

Examples:

- belian categories
- quasi-abelian categories

Neither notion is self-dual, e.g. the opposite of a belian category need not admit a belian structure. In particular, these notions are *not* well-adapted to Grothendieck–Witt theory, which takes as input an *exact* functor $P : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$.

Proto-exact categories (after Dyckerhoff–Kapranov)

This is the data of

- a pointed category $\mathcal{A}$
- distinguished classes of morphisms $\mathfrak{I}$ and $\mathfrak{D}$, denoted by $\hookrightarrow$ and $\twoheadrightarrow$.

This data must satisfy a number of self-dual axioms, the most important of which (for today's purposes) states:

$$\begin{tikzcd}
U \times_V W \arrow[r, two heads, dashed] \arrow[d, hook, dashed] & W \arrow[d, hook] \\
U \arrow[r, two heads] & V
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
V \arrow[r, two heads] \arrow[d, hook] & W \arrow[d, hook, dashed] \\
U \arrow[r, two heads, dashed] & U \sqcup_V W
\end{tikzcd}$$

Basic observations

- Quillen's Q-construction extends directly to proto-exact categories.
- The opposite of a proto-exact category is proto-exact.

Key local example

- Let A be a commutative pointed monoid
- Let $\mathcal{A} = A$-proj, fg projective A-modules
- A morphism $f : U \to V$ in $\mathcal{A}$ is called *normal* if $f^{-1}(v)$ is empty or a singleton whenever $v \neq 0_V$.
- $\mathcal{A}$ has a proto-exact structure in which
  - $\hookrightarrow =$ injective A-module homomorphism
  - $\twoheadrightarrow =$ surjective normal A-module homomorphism
- Additional structure: $\mathcal{A}$ has an exact symmetric monoidal structure $\bigoplus$ so that, for example, there exist split exact sequences

$$U \hookrightarrow U \bigoplus V \twoheadrightarrow V.$$

  In fact, every short exact sequence in $\mathcal{A}$ is *uniquely* isomorphic to one of this form.

- For later use, let A-proj$^{\text{nor}} \subset A$-proj be the (proto-exact) subcategory of normal morphisms

Previous results: K-theory over $\mathbb{F}_1$ and homotopy theory

A pointed monoid A is called *cancellative* if

$$ab = ac \implies ab = ac = 0 \text{ or } b = c.$$

Theorem (Deitmar, Chu–Morava; see also [ELY])

Let A be cancellative. There are weak equivalences

$$\mathcal{K}(A\text{-proj}^{\text{nor}}) \simeq \mathcal{K}(A\text{-proj}) \simeq \mathbb{Z} \times B(A^\times \wr \Sigma_\infty)^+.$$

Note: The proof depends on Quillen's $Q = +$ theorem.

Example: Let $A = \mathbb{F}_1 = \{0, 1\}$, so that $A^\times = \{1\}$. Then

$$\mathcal{K}(\mathrm{Vect}_{\mathbb{F}_1}) \simeq \mathbb{Z} \times B\Sigma_\infty^+ \overset{BPQ}{\simeq} \Omega^\infty S^\infty$$

so that $K_i(\mathrm{Vect}_{\mathbb{F}_1}) \simeq \pi_i^s$.

Example of similar flavour (Eppolito–Jun–Szczesny): There exists a map $\Omega^\infty S^\infty \to \mathcal{K}(\mathrm{Matroids})$ which is injective on $\pi_i(-)$.

K-theory of monoid schemes

We work with monoid schemes as in Samarpita's talk.

Definition

Let $\mathrm{Vect}(X) \subset \mathrm{Coh}(X)$ be the subcategory of (finite rank) vector bundles and their *normal* $\mathcal{O}_X$-module homomorphisms.

Proposition ([CLS]; see also [Jun–Szczesny], [ELY])

The category $\mathrm{Vect}(X)$ is proto-exact. Moreover, $\mathrm{Vect}(X)$ is uniquely split and combinatorial.

That $\mathrm{Vect}(X)$ is proto-exact allowed [CLS] to define and study its algebraic K-theory groups $K_i(X)$.

The *combinatorial* notion is motivated by work of Szczesny. Roughly, it states that if $i : U \hookrightarrow V \oplus W$, then

$$U \simeq (U \cap V) \oplus (U \cap W)$$

compatibly with i.

Our approach

Use *rigidity* in $\mathbb{F}_1$-geometry to suitably decompose $\mathrm{Vect}(X)$, and interpret this decomposition in K-theory.

A pointed commutative monoid is called *reversible* if any two prime ideals intersect non-trivially.

Globalize the notions of cancellative and reversible to monoid schemes stalkwise and locally, respectively.

Lemma ([ELY])

A monoid scheme X is integral if and only if it is irreducible, reversible and (partially) cancellative.

- Let $\mathrm{Pic}(X)$ be the Picard group of X
- Let $\langle \mathcal{O}_X \rangle \subset \mathrm{Vect}(X)$ be the full subcategory on $\mathcal{O}_X^{\oplus n}$, $n \geqslant 0$; it is a proto-exact subcategory stable under $\oplus$ and $\otimes_{\mathcal{O}_X}$
- Form the "group algebra of $\mathrm{Pic}(X)$ with coefficients in $\langle \mathcal{O}_X \rangle$":

$$\langle \mathcal{O}_X \rangle [\mathrm{Pic}(X)]$$

Proposition ([ELY])

Let X be an integral monoid scheme.

1. The functor $\Gamma(-, X) : \langle \mathcal{O}_X \rangle \to \Gamma(X)\text{-proj}^{\mathrm{nor}}$ is an exact equivalence.
2. The functor

$$\langle \mathcal{O}_X \rangle [\mathrm{Pic}(X)] \to \mathrm{Vect}(X), \qquad \bigoplus_{\mathcal{L} \in \mathrm{Pic}(X)} \mathcal{E}_{\mathcal{L}} \mapsto \bigoplus_{\mathcal{L} \in \mathrm{Pic}(X)} \mathcal{E}_{\mathcal{L}} \otimes_{\mathcal{O}_X} \mathcal{L}$$

induces a homotopy equivalence on K-theory spaces.

Theorem ([ELY])

For an integral monoid scheme X, there is a homotopy equivalence

$$\mathcal{K}(X) \simeq \prod_{\mathcal{L} \in \mathrm{Pic}(X)} \mathcal{K}(\Gamma(X)\text{-proj}).$$

Proof:

$$\begin{aligned} \mathcal{K}(X) &\simeq \mathcal{K}(\langle \mathcal{O}_X \rangle [\mathrm{Pic}(X)]) \\ &\simeq \prod_{\mathcal{L} \in \mathrm{Pic}(X)} \mathcal{K}(\langle \mathcal{O}_X \rangle_{\mathcal{L}}) \\ &\simeq \prod_{\mathcal{L} \in \mathrm{Pic}(X)} \mathcal{K}(\Gamma(X)\text{-proj}). \end{aligned}$$

Corollary ([CLS])

There is a ring isomorphism $K_0(X) \simeq \mathbb{Z}[\mathrm{Pic}(X)]$.

Example: $X = \mathbb{P}^n_{\mathbb{F}_1}$

There is a group isomorphism (Hüttemann, [CLS], Flores–Weibel)

$$\mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(\mathbb{P}^n_{\mathbb{F}_1}), \qquad m \mapsto \mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(m).$$

So, for example,

$$K_0(\mathbb{P}^n_{\mathbb{F}_1}) \simeq \mathbb{Z}[\mathbb{Z}].$$

In particular, $\mathcal{K}(\mathbb{P}^n_{\mathbb{F}_1})$ is independent of n. This is in stark contrast to the case over fields, where this a relation in degree $n+1$ (Quillen).

Grothendieck–Witt theory of monoid schemes

Constructing dualities

Basic problem (already locally)

Let A be a pointed commutative monoid. The functor $\mathrm{Hom}_A(-, A)$ does *not* define a duality on A-proj.

For example, we have

$$\mathrm{Hom}_A(A^{\oplus n}, A) \simeq A^{\times n}.$$

However, we have the following partial fix.

Lemma ([ELY])

Let A be cancellative and reversible. Then $\mathrm{Hom}_A^{\mathrm{nor}}(-, A)$ is an exact duality on $A\text{-proj}^{\mathrm{nor}} \subset A$-proj.

This previous local example globalizes:

Lemma ([ELY])

Let X be an integral monoid scheme. Then $\mathrm{Hom}^{\mathrm{nor}}_{\mathcal{O}_X}(-,\mathcal{O}_X)$ is an exact duality on $\mathrm{Vect}(X)$.

- There is an induced set theoretic involution of $\mathrm{Pic}(X)$, so that

$$\mathrm{Pic}(X) = \mathrm{Pic}(X)^{\mathbb{Z}_2} \sqcup \mathrm{Pic}(X)^*$$

- Since $\langle \mathcal{O}_X \rangle$ is stable under duality, there is an induced involution of $\langle \mathcal{O}_X \rangle[\mathrm{Pic}(X)]$. To understand this involution, there are two cases to consider:
  - If $\mathcal{L} \in \mathrm{Pic}(X)$ is self-dual, then $\langle \mathcal{O}_X \rangle_{\mathcal{L}}$ is stable under duality.
  - If $\mathcal{L} \in \mathrm{Pic}(X)$ is not self-dual, then $\langle \mathcal{O}_X \rangle_{\{\mathcal{L},\mathcal{L}^\vee\}}$ is stable under duality; it is the *hyperbolic category* on $\langle \mathcal{O}_X \rangle$. We have

$$\mathcal{GW}(\langle \mathcal{O}_X \rangle_{\{\mathcal{L},\mathcal{L}^\vee\}}) \simeq \mathcal{K}(\langle \mathcal{O}_X \rangle).$$

The above considerations lead to the following result.

Theorem ([ELY])

For an integral monoid scheme X, there is a weak homotopy equivalence

$$\mathcal{GW}(X) \simeq \prod_{\mathcal{L} \in \mathrm{Pic}(X)^{\mathbb{Z}_2}} \mathcal{GW}(\Gamma(X)\text{-proj}^{\mathrm{nor}}) \times \prod_{\mathcal{L} \in \mathrm{Pic}(X)^* / \mathbb{Z}_2} \mathcal{K}(\Gamma(X)\text{-proj}).$$

Example: $X = \mathbb{P}^n_{\mathbb{F}_1}$. The involuation on $\mathbb{Z} \simeq \mathrm{Pic}(\mathbb{P}^n_{\mathbb{F}_1})$ is negation, so that

$$\mathcal{GW}(\mathbb{P}^n_{\mathbb{F}_1}) \simeq \mathcal{GW}(\mathrm{Vect}_{\mathbb{F}_1}) \times \prod_{m>0} \mathcal{K}(\mathrm{Vect}_{\mathbb{F}_1}).$$

Our problem is therefore reduced to the local case: describe

$$\mathcal{GW}(\Gamma(X)\text{-proj}^{\text{nor}}).$$

Technical problem: Quillen's $Q = +$ theorem does *not* hold for Grothendieck–Witt theory over $\mathbb{F}_1$.

Instead, we have the following result, which holds more generally for combinatorial unqiuely split proto-exact categories.

Theorem ([ELY])

There is a weak homotopy equivalence

$$\mathcal{GW}(\Gamma(X)\text{-proj}^{\text{nor}}) \simeq \bigsqcup_{w \in W_0} B\text{Isom}(s) \times \mathcal{GW}_H^{\oplus}(\Gamma(X)\text{-proj}^{\text{nor}}).$$

Example: $\Gamma(X) = \mathbb{F}_1$. Then

$$\mathcal{GW}(\text{Vect}_{\mathbb{F}_1}) \simeq \bigsqcup_{m \geqslant 0} B\Sigma_m \times \mathbb{Z} \times B(\mathbb{Z}_2 \wr \Sigma_\infty)^+.$$

Application: Projective bundle theorems

The main technical result is a direct analogue of a well-known result over fields; the proof, however, is different.

Proposition ([ELY])

Let $\mathcal{V} \to X$ be a vector bundle over an integral monoid scheme. Then the map

$$\mathrm{Pic}(X) \times \mathbb{Z} \to \mathrm{Pic}(\mathbb{P}\mathcal{V}), \qquad (\mathcal{L}, n) \mapsto \pi^* \mathcal{L}(n)$$

is a $\mathbb{Z}_2$-equivariant group isomorphism.

Theorem ([ELY])

Let $\mathcal{V} \to X$ be a vector bundle over an integral monoid scheme. There is a homotopy equivalence

$$\mathcal{K}(\mathbb{P}\mathcal{V}) \simeq \prod_{\mathbb{Z}} \mathcal{K}(X).$$